A. R. Wulsin

MATHEMATICAL ANALYSIS
HIGHER COURSE

By

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An Introduction to Mathematical Analysis

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EDITOR'S INTRODUCTION

Four years ago, in an address before the Mathematical Association of America, I called attention to the need of a text to serve for the second year of collegiate instruction in mathematics in those institutions that give the "new type" of course in the freshman year. It is only natural, therefore, that I feel especially gratified in seeing this new volume by Professor Griffin added to the series of texts appearing under my editorship.

The marked success attained by Professor Griffin's *Introduction to Mathematical Analysis* makes it appropriate that he should be the first to publish a *Higher Course.* I feel confident that the many enthusiastic users of the *Introduction,* who have been expectantly awaiting this second volume, will not be disappointed. It would seem altogether feasible, however, for others as well to make use of the present volume. The "new type" of course for freshmen, while by no means as yet standardized, does contain a fairly well-defined body of material, notably a substantial introduction to the concepts and elementary methods of the differential and integral calculus. Any student who has had such an introduction in his freshman year (whether he used Professor Griffin's earlier text or not) could presumably use the present text with profit in his sophomore year. This is due to the author's extended summaries and reviews of the earlier work placed at appropriate places in the new volume. As he says in his Preface a little extra time spent on these summaries and reviews would overcome any lack of the necessary articulation between the present text and the work of the freshman year, in those cases where Professor Griffin's *Introduction* was not used.

The author has, in his Preface, called attention to several features of the present volume. There is then no need to repeat them here. Suffice it to say that, in my opinion, Professor Griffin has carried out admirably the purposes which such a second course should serve: to carry forward the mathematical training begun in the first year, with special reference to the needs of those students who need a working knowledge of mathematical methods in their professions or vocations. The acquisition of a good technique, which properly received less emphasis in the first course, is made a primary purpose of the present one. And, as Professor Griffin says, such a technique is far more readily acquired on the basis of a previous acquaintance with fundamental concepts than when the attempt is made without an adequate basis of this sort. On the other hand he has skillfully avoided the pitfall of presenting the drill necessary to acquire this technique without the enlivening presence of new and interesting applications.

In my opinion this second volume will equal or even surpass the first in usefulness.

J. W. Young

PREFACE

THE generous reception accorded the author's *Introduction to Mathematical Analysis* has created a need for an additional volume to serve as a text for the following year. It is rather generally recognized that non-specialist students, who take but a single year of college mathematics, are best served by a unified course of broad scope. There has, however, been some discussion as to whether such a unified course, or a series of separate specialized courses, will afford the better foundation for those students who are to continue their mathematical study, — especially for those who expect to use mathematics extensively in their life work, and who need to gain a good command of technique. Manifestly the answer to this question depends not only upon the introductory course, but also upon the subsequent instruction. Any potential values of a unified freshman course as a foundation for later study can be realized in full only if the work of the second year is based definitely upon that of the first, utilizing its elements of strength and supplementing it where not yet adequate to the larger purpose.

The present volume is systematically built upon and correlated with the author's earlier text, and will suggest what may be expected from a unified two-year course, begun with the preceding volume. Long experience indicates that a thorough trial will clearly show such a unified course to be superior to separate specialized courses, for students of engineering, pure science and mathematics as well as for others.

It is in fact, something of a revelation to observe the degree of facility which students develop in using the calculus, and

the insight which they acquire into its significance, when it is kept before them continuously for two years under a consistent plan and is applied to a sufficiently wide range of problems. Nor are the other mathematical subjects neglected, as each is frequently drawn upon during the two years, all being closely interwoven.

Like its predecessor the present volume embodies a course evolved and taught in Reed College. Fourteen years of careful experimentation and trial have served to test adequately the teachableness and value of the subject matter here presented. Quite apart from any question of unified courses or specialized courses, it appears desirable to incorporate in a mathematical text some of the materials and methods used here. Attention is called especially to a number of features:

1. The opening chapter, giving a concise treatment of principles needed as tools in calculus problems, and the homogeneity test for the correctness of equations.

2. The method of finding intersections of surfaces graphically, and actually *seeing* the limits of integration in problems often regarded as troublesome. Note also the prominence of problems involving the familiar elementary surfaces, with no equations given.

3. The treatment of the technique of integration. Instead of devoting the entire time to showing students how to integrate without the help of tables, or throwing them wholly upon tables, a systematic plan is worked out to develop familiarity with the best final procedure: when to use tables and when not.

4. The systematic tests for the consistency of two supposed integrals of a common integrand.

5. The attention given to mean values, approximate integration, and improper integrals. Note the careful definition of a definite integral as the limit of a sum, as a basis for critical

discussions; also the elementary use of finite differences in connection with functions defined by tables.

6. The three-fold treatment of locus problems: by experimental constructions, by deriving and *interpreting* the equations, and by direct geometrical proofs.

7. The discussion of the solutions of differential equations in concrete problems.

8. The minimum use of rules and formulas, as for instance in rationalizing binomial radical forms, reading off subtangents from rectangular or polar figures, etc.

9. The unusual range of applications, not only in the engineering field and all branches of natural science, but also in economics and actuarial science, without presupposing a knowledge of these fields.

10. The frequent recurrence of basic ideas; *e.g.*, the treatment of centroids, etc., by single, double, and triple integration, until the idea is an old story; or, again, the review problems on technique while dealing with applications of integration, and vice versa.

11. The summaries of principles touched upon in the earlier course. These serve to make a close correlation with that course, and clinch the ideas introduced there. They also enable students who have had a different but approximately equivalent introductory course to handle this one satisfactorily by spending a little extra time on these sections.

12. The order of topics. This places the work in integration early enough to give it the maximum utility in other scientific courses, and also let it be employed to advantage in some of the later chapters of the differential calculus. The whole constitutes a highly organized *course*. Nevertheless, it is flexible in certain ways, as mentioned below.

In the course as given at Reed College, Chapters I–V are covered during the first semester, and the others during the

second. In recent years, however, we have postponed Part II of Chapter X to a later course (Higher Geometry, Analytic and Synthetic); and have devoted the time thus released to more drill work on the other portions of this course. An abundance of exercises is provided, so that, for the most part, the same problems need not be assigned for home work in consecutive years. Doubtless, however, some teachers will prefer to cover in the sophomore year the geometrical properties of conics treated in Part II of Chapter X; and the material is available for that purpose. Some may decide to omit Part II or III or IV of Chapter VI, or the latter portions of Chapters IX, XI, and XII, or the more specialized and technical applications in other chapters. These omissions can be made without confusion. Thus the course is adaptable to varied conditions.

The applications have been drawn from or inspired by many sources. To give a complete list of them is impracticable; to give a partial one would involve invidious distinctions. To all I would express my gratitude.

I am indebted to Professor J. W. Young for several very valuable suggestions, and to my colleague Professor Jessie M. Short for help in reading the galley proofs and for other assistance. It is also a pleasure to mention the skillful and discerning work done upon the mechanical drawings by two of my students, Messrs. S. W. Nile, Jr., and T. C. Roake. And I welcome this opportunity of expressing my appreciation of the coöperation of the publishers, both on the earlier volume and upon this one.

F. L. Griffin

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MATHEMATICAL ANALYSIS HIGHER COURSE

FOREWORD TO STUDENTS

(I) **Mathematics and Human Progress.** Mathematics has contributed largely to human progress, both intellectual and material. Its study of the foundations of knowledge is basic for modern thought. Its methods of systematic analysis and calculation have added much to man's understanding and mastery of his physical environment.

Mathematical Analysis deals primarily with *relations among varying quantities*, — with the changes in one quantity which will result from specified changes in some other quantity or quantities. It develops methods of discovering laws and of predicting future changes. Thus it makes possible an effective study of such diverse matters as the attractions of the heavenly bodies and the resulting movements of the planets, the cooling of the earth's crust, the progress of chemical reactions, the growth and senescence of organisms, the oscillations of electric currents and radio waves, the flow of liquids and gases, the flight of airplanes and projectiles, the proper design for a bridge or reservoir, and a host of related topics in engineering, actuarial science and investments, medicine, physiology, psychology, and music.

(II) **Aim and Scope of This Course.** Some of the scientific problems just mentioned are touched upon briefly in the

introductory course which many of you have already taken.* Some other topics are highly technical and require more extensive mathematical equipment than can be provided here. We shall, however, cover the fundamental principles and technique of mathematical analysis and their basic applications to the various sciences. This will afford an adequate foundation for working ahead in any field, — both as to the necessary facility in performing fundamental operations and as to the ability to apply the principles to new subject matter.

To ensure continuity and completeness of treatment, and for the information of students who have taken a different earlier course, some topics which were covered in the *Introduction* will be reviewed or summarized briefly here. Make sure that you have these clearly in mind; but give particular attention to the important new formulas and principles. It will be helpful to *keep a list* of these latter in your notebook, in some place reserved for the purpose. Add each new formula as you come to it; and *look the list over, frequently,* to fix the ideas in mind. Above all, be clear as to what each new technical word *means,* and as to the general plan or *mode of procedure* in solving problems of a given type.

* F. L. Griffin, *An Introduction to Mathematical Analysis.* This will be referred to throughout the present text simply as "*Introduction*" or "*Intro.*"

CHAPTER I

CURVES AND THEIR EQUATIONS

PART I. RECTANGULAR COÖRDINATES

§ 1. Summary of Familiar Principles. The curves which are encountered in scientific problems can be studied by means of coördinates and equations. It is essential to master a number of principles and formulas for use as working tools. The following brief summary will recall some fundamental ideas already familiar.

(*A*) *Position of a Point.* The location of a point in a plane may be described by its abscissa x and ordinate y, referred to a pair of mutually perpendicular axes in the plane.

The plane may have any direction in space; but for brevity we shall often refer to a line in the X-direction as horizontal and to one in the Y-direction as vertical.

(*B*) *Basic Formulas.* The distance between two points, and the slope and midpoint of the line joining them are:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}, \tag{1}$$

$$l=\frac{y_2-y_1}{x_2-x_1}, \tag{2}$$

$$\bar{x}=\tfrac{1}{2}(x_1+x_2), \qquad \bar{y}=\tfrac{1}{2}(y_1+y_2). \tag{3}$$

Two lines with slopes l_1 and l_2 are

parallel, if $$l_1=l_2;$$

perpendicular, if $$l_1 l_2=-1, \quad \text{or} \quad l_2=-\frac{1}{l_1}. \tag{4}$$

(*C*) *Equation and Locus.* An equation which is satisfied at every point (x, y) of a given curve, and at no other points, is called "the equation of the curve." Conversely, all points whose coördinates satisfy a given equation constitute "the locus of the equation." We shall consider only real loci, made up of points with real coördinates.

To test whether a given point lies on a curve, substitute its coördinates in the equation of the curve and see whether the equation is satisfied. A "curve" may be a straight line as a special case.

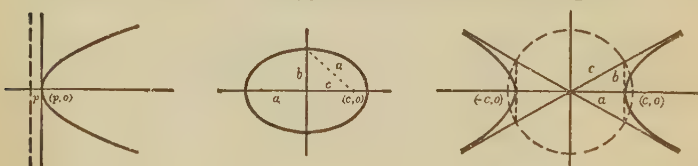
Every curve that can be drawn has an equation; but not every equation has a real locus. *E.g.*, $x^2+y^2+1=0$ has none. (Why not?) Sometimes the locus consists of a few isolated points rather than a curve. Thus the locus of $(x-3)^2(x-4)^2+(y-9)^2=0$ is the pair of points (3, 9) and (4, 9). (Verify each. Why no others?)

(*D*) *Standard Equations.* Every straight line has a linear equation; and every linear equation represents some straight line. For certain other common loci the equations are:

Circle with center (h, k) and radius a:

$$(x-h)^2+(y-k)^2=a^2. \tag{5}$$

Parabola, *ellipse*, and *hyperbola*, as shown in Fig. 1:



$y^2 = 4px$ $\qquad$ $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ $\qquad$ $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

Fig. 1.

(*E*) *Displacements.* Shifting a curve h units to the right and k units upward, without rotation, replaces

$$x \text{ by } (x-h), \qquad y \text{ by } (y-k),$$

everywhere in the equation of the curve. (*Intro.*, § 222.)

Rotating a curve $+90°$, without translation, replaces every

$$x \text{ by } y, \qquad y \text{ by} -x.$$

(Other amounts of rotation will be discussed later.)

(*F*) *Intersections; Intercepts.* To find the intersection of two curves, solve their equations simultaneously. In particular, to find the intersection of a curve with the X-axis, put $y = 0$ in the equation and solve for x. The values of x so obtained are called the *intercepts* on the X-axis. Similarly for the Y-axis. If the intercepts are imaginary, the curve does not meet the axes.

To plot a curve by points we usually give values to x and solve for y, or *vice versa*. This amounts to finding points on the curve where it crosses various vertical lines, $x=1, 2, 3$, etc., or various horizontal lines.

In solving two equations simultaneously we combine them so as to get simpler equations. This amounts to replacing the given curves by more convenient curves or lines through the same intersections. *E.g.*, for the circles

$$x^2+y^2=25, \qquad x^2+y^2+2\,x+4\,y=44,$$

we would first subtract one equation from the other, getting

$$2\,x+4\,y=19.$$

This last represents some straight line. And that line must pass through the intersections of the two circles; for the last equation is satisfied at any points where both given equations are satisfied.

(*G*) *Infinity; Asymptotes.* It may happen that, as x approaches some value a, y increases without limit; but, when $x=a$, y ceases to exist. (This will be the case, for instance, if y equals some fraction whose denominator becomes zero at $x=a$, while the numerator takes some other value.)

There is then no point on the curve at which $x=a$; but the vertical line at $x=a$ is an asymptote approached closely by the curve as the latter rises or falls without bound.

To cover all this we shall usually say briefly:
y becomes infinite, or equals infinity, at $x=a$; written

$$y=\infty \text{ at } x=a.$$

This must be understood always in the sense mentioned above, and not construed literally to mean that y equals some enormous value called "infinity." There is no such number or value!

Again, at a point where the tangent to a curve is vertical, we shall call the slope infinite, meaning:

(1) There is no such thing as the slope of a vertical line;

(2) The slope of the curve nearby is very great, and increases without limit as we approach the point in question.

We locate infinite values of any quantity by finding where its reciprocal is zero. Thus

$$\csc\theta=\infty \qquad \text{where } \sin\theta=0;$$

$$\frac{60}{x-2}=\infty \qquad \text{where } x-2=0.$$

If we wish to note the fact that some quantity Q remains *negative* while becoming infinite numerically, we write

$$Q=-\infty.$$

EXERCISES

1. Find the distance between $(3, -1)$ and $(11, 5)$ by formula; also the midpoint, and the slope of the joining line. Check by plotting.

2. The same as Ex. 1 for the following pairs of points:

(*a*) $(2, 5)$, $(5, 1)$; (*b*) $(-15, -25)$, $(-3, -20)$.

3. The vertices of a triangle are $(-4, -1)$, $(2, 7)$, and $(18, 19)$. Show by calculation that the line joining the midpoints of some pair of sides is parallel to the other side and equal to half of it.

4. The same as Ex. 3 for the triangle $(7, -4)$, $(31, -14)$, $(21, 10)$.

5. Is the line joining $(10, 1)$ and $(4, 9)$ perpendicular to the line in Ex. 1?

6. In Ex. 4 find the slope of the median drawn from $(31, -14)$; also the slope of the opposite side. Discuss.

7. Which of the points $A(15, -20)$, $B(24, 7)$, $C(-16, -19)$, $D(22, 12)$ are on the curve $x^2+y^2=625$? Verify that the lines joining either to the ends of the horizontal diameter are perpendicular.

8. Recognize and draw freely the loci of

(*a*) $x^2+y^2-8x+4y=11$, (*b*) $y^2=-12x$,
(*c*) $(x+2)^2=4(y-6)$, (*d*) $16x^2-9y^2=144$,
(*e*) $\dfrac{(x-5)^2}{36}+\dfrac{y^2}{100}=1$, (*f*) $\dfrac{y^2}{64}-\dfrac{x^2}{36}=1$.

9. Write the equation of a circle with

(*a*) center $(-2, 5)$, radius 7; (*b*) center $(4, 0)$, radius 2;
(*c*) center at $(3, -2)$ and passing through $(5, 7)$;
(*d*) ends of one diameter at $(8, -1)$ and $(-2, 11)$.

10. Show that the locus of each following equation consists of an isolated point, or points:

(*a*) $x^2+(2y-5)^2=0$; (*b*) $(x-2)^2(x+3)^2+(y-7)^4=0$.

11. In each of the following equations does y become infinite for some value of x? If so, for what value?

(*a*) $y=\dfrac{7x+2}{3x-8}$; (*b*) $y=\dfrac{6x}{x^2+4}$;
(*c*) $y=\dfrac{e^{4x}}{e^x-e}$; (*d*) $y=\dfrac{x+4}{\log x}$;
(*e*) $xy=20$; (*f*) $\sqrt{x-2}\,(y+5)=80$.

12. Find the intercepts of each of the following curves:

(*a*) $x^2+y^2+6x-8y=0$; (*b*) $4x^2+6xy-2y^2-12=0$;
(*c*) $x^2+8x+4y+20=0$; (*d*) $y+2x^3+7x^2-9=0$.

13. Find the intersections of

(*a*) $y^2=9x$ with $2x-y=2$;
(*b*) $x^3+y^3=6xy$ with $y=x$; also with $y=2x$;
(*c*) $x^2+y^2=32$ with $y^2=x^2(5+x)$.

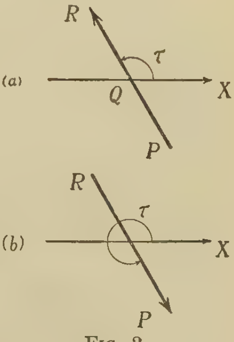


FIG. 2.

§ 2. Direction Angle. To describe any *direction* in the XY-plane, — such as the direction from P to R in Fig. 2 *a*, — we may give the angle XQR, running counterclockwise from the positive X-axis around to that direction. This angle is denoted by τ

(Greek letter *tau*); and is called the "direction angle" for the directed line PR.

Observe from Fig. 2 *b*, that, for the directed line RP going *from* R *to* P, angle τ would be larger by 180°.

The *inclination* of a line, as we already know, is the angle I at which the line rises or falls, going toward the right. Thus, for the line joining P and R in either Fig. $2a$ or $2b$, I is a negative acute angle. Always, I lies between $-90°$ and $+90°$.

Angles τ and I, while often different, are always related. *E.g.*, if $I=-60°$, τ must be 120°, or else 300°; or one of these plus some multiple of 360°.

Clearly $\sin\tau$ and $\cos\tau$ may be opposite in sign to $\sin I$ and $\cos I$. But $\tan\tau$ and $\tan I$ are always equal, and either equals the slope of the line:

$$\tan\tau=\tan I=l. \tag{6}$$

By (6) we can find τ or I if the slope l is known, or vice versa. (For a vertical line: $\tau=90°$ or 270°, $I=\pm 90°$, $l=\pm\infty$.)

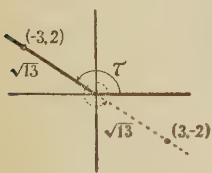


Fig. 3.

Further, if l or $\tan\tau$ is known, $\sin\tau$ and $\cos\tau$ can be read off from a rough figure. (Cf. *Intro.*, § 255.)

E.g., if $\tan\tau=-\frac{2}{3}$, Fig. 3 shows that either

$$\sin\tau=\frac{2}{\sqrt{13}},\qquad \cos\tau=-\frac{3}{\sqrt{13}},$$

or else

$$\sin\tau=-\frac{2}{\sqrt{13}},\qquad \cos\tau=\frac{3}{\sqrt{13}}.$$

§ 3. Equations of a Straight Line. For a line through a given point (x_1, y_1) with a slope l, the equation is, by *Intro.*, § 226:

$$y-y_1=l(x-x_1). \tag{7}$$

For a non-vertical line through *two given points*, we first find l by (2) and then use (7), considering either given point as (x_1, y_1).

For a vertical line, $l=\pm\infty$, and we cannot use (7). But evidently $x=x_1$ all along the line, and this is the equation of the line.

If a line has an intercept b on the Y-axis, we have $x_1=0$, $y_1=b$; and equation (7) becomes:

$$y=lx+b. \tag{8}$$

If a line, not passing through the origin, has intercepts a and b on the X- and Y-axes respectively, its slope is $-b/a$; and (8) becomes

$$y=-\frac{b}{a}x+b.$$

Transposing the x-term and dividing through by b:

$$\frac{x}{a}+\frac{y}{b}=1. \tag{9}$$

By (8) and (9) we can immediately write the equation of a straight line if given its slope and Y-intercept or its two intercepts.

To find the slope of a non-vertical line when given its equation, simply solve the equation for y, getting the form (8): the coefficient of x will be the slope l. (*Intro.*, § 200.)

EXERCISES

1. Find the slope of a straight line for which the direction angle is $\tau=140°$; $\tau=260°$; $\tau=315°$. What is l for each?

2. If the slope of a straight line is $-\frac{4}{3}$, what is l? What values may τ have? $\sin\tau$? $\cos\tau$?

3. The same as Ex. 2 if the slope is $\frac{12}{5}$?

4. Going around the circle $x^2+y^2=25$ counterclockwise, what is τ for the direction of motion when passing through (3, 4)? Through (−4, 3)? What is l for the tangent line at each of these points?

In Exs. 5–9, write and simplify the equation of the straight line which meets the specified requirements.

5. The line through the point named, with the given slope:

(*a*) (2, 5), slope 4; (*b*) (7, −1), slope $-\frac{2}{3}$;

(*c*) (8, 15), slope $-\frac{8}{15}$; (*d*) $(\sqrt{75}, -5)$, slope $\sqrt{3}$.

6. The line through the two given points:

(*a*) (1, 4), (7, 6); (*b*) (−2, 5), (4, −1).

7. (*a*) The line which cuts the parabola $y=x^2$ at $x=-1$ and at $x=2$. (*b*) The line which cuts $y^2=4x$ at $y=-1$ and at $y=36$.

8. The line with intercepts, etc., as shown:

(a) $b=8$, $l=-2$; (b) $b=-3$, $\tau=135°$;
(c) $a=12$, $b=6$; (d) $a=-5$, $b=5$.

9. The diagonals of a rectangle with base 12 and height 8, if the base and left end are used as X- and Y-axes. The same if the base is 4 and the height 6.

10. Find the slope and inclination for each of the following lines; also the direction angle, considering the line as running toward the right.

(a) $2x+5y=20$, (b) $x-y=12$, (c) $\sqrt{3}x+3y=5$.

11. Is the line $5x-3y+30=0$ perpendicular to the line from (12, −2) to (15, −4)? Is it perpendicular to the line which joins the center of $x^2+y^2-6x+8y=0$ to the midpoint of (7, −25) and (19, 5)? Reasons?

12. Find the equations of two straight lines through (7, 2): one parallel to the line $2x-3y=6$ and the other perpendicular thereto.

§ 4. **Reflection.** In recognizing the character of loci, it is helpful to know the effect of making certain simple changes or substitutions in an equation.

(*A*) *Changing the sign of either x or y.* If in an equation every x be replaced by $-x$, the curve will be *reflected* in the Y-axis. That is, any point P will be moved to a new position P' symmetric to P with respect to OY.

Thus, $y^2=4px$ is a parabola extending to the right of OY, while $y^2=-4px$ is its reflection extending to the left.

Similarly, replacing y by $-y$ reflects a curve in the X-axis.

(*B*) *Changing the signs of both x and y.* This reflects the curve in both axes, and hence through the origin, every point (x_1, y_1) being moved to the opposite point $(-x_1, -y_1)$.

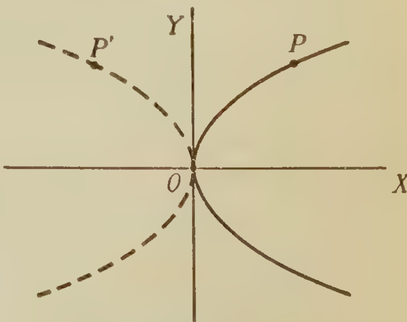


FIG. 4.

(*C*) *Interchanging x and y.* This reflects the curve in the line $y=x$ which bisects the first and third quadrants.

Ex. I. If the exponential curve $y=e^x$ be reflected in the line $y=x$, what is the resulting curve?

Interchanging: $x=e^y$.
Solving for y: $y=\log x$.
Hence the reflected curve is the graph of $\log x$, base e. (Fig. 5.)

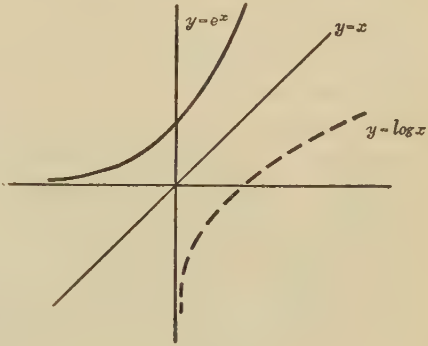


FIG. 5.

§ 5. Symmetry. If any equation is unaltered by one of the operations (*A*), (*B*), (*C*) of § 4, the curve is the same after reflection as it was before. That is, the original curve was symmetrical in some way.

Equation unchanged by	*Curve symmetrical as to*
Replacing x by $-x$	Y-axis
Replacing y by $-y$	X-axis
Both substitutions above	Origin
Interchanging x and y	Line $y=x$.

Ex. I. The ellipse $$\frac{x^2}{100}+\frac{y^2}{64}=1.$$

This is symmetrical with respect to each axis and the origin, but not with respect to the line $y=x$.

Ex. II. The rectangular hyperbola $xy=50$.

This is symmetrical with respect to the line $y=x$ and the origin, but not with respect to either axis. (Cf. *Intro.*, §§ 220, 221.)

§ 6. Homogeneity Test. The letters used in mathematical formulas stand for *numbers*. When we speak of "the distance s," we mean that s is the *number* of feet or inches, etc.

If we write $s=20\,t$, we do not mean that multiplying time by the number 20 will somehow give a distance. We mean that the number of seconds, multiplied by 20 (which, let us say, is the number of feet traveled per second), will give the number of feet traveled in any time.

If, instead, we wrote $s=vt$, where v denotes the constant speed, it would be clear that the right-hand member as well as the left would denote some distance.

In a *general* formula in which every geometrical or physical magnitude involved is represented by a letter, rather than by an absolute numerical value such as 20 above, each term in the equation will be clearly of the same kind or " dimensions." If one complete term represents a distance, so must every term.

Whenever we start with such a general equation and make any general calculation, the equation at every step must remain *homogeneous*, — *i.e.*, having all its terms of like kind. An error can often be detected by noting at what stage the homogeneity disappears.

To illustrate more concretely, consider the equation of the standard ellipse in the form

$$b^2x^2+a^2y^2=a^2b^2. \tag{10}$$

Here every term is formed by multiplying four numbers (such as b, b, x, x), each designating a number of distance-units. *Symbolically expressed*, the equation is of the form

$$(\text{distance})^4+(\text{distance})^4=(\text{distance})^4$$

and is thus homogeneous.

If the ellipse were moved k units upward, its equation would become

$$b^2x^2+a^2(y-k)^2=a^2b^2, \tag{11}$$

which is still homogeneous, of degree 4. If in going from (10) to (11) we had carelessly dropped the exponent of b on the right side, the error would be obvious on inspection, as (11) would then be symbolically *of the impossible type:*

$$(\text{distance})^4+(\text{distance})^4=(\text{distance})^3.$$

Terms like x^2/a^2, being of the type $(\text{distance})^2 \div (\text{distance})^2$, are regarded as $(\text{distance})^0$; terms like $\sqrt{x^2+y^2}$, as $(\text{distance})^1$; terms like $(a^2-b^2)^{\frac{5}{2}}$, as $(\text{distance})^5$; etc. The slope of a line is $(\text{distance})^0$ or a pure number; so is any trigonometric function. Neither affects the type of a term in which it appears.

Ex. I. The radius of a circle associated with the ellipse $b^2x^2+a^2y^2=a^2b^2$ was once carelessly calculated as

$$R=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{\dfrac{a^2b^4}{y}},$$

no numerical substitutions having been made. Show this result impossible.

Here the numerator is of the type $[(\text{distance})^6]^{\frac{3}{2}}$, *i.e.*, $(\text{distance})^9$; the denominator, $(\text{distance})^5$; and hence the fraction, $(\text{distance})^4$. This cannot equal a simple distance R.

EXERCISES

1. What if any symmetry has each of these loci?

(a) $x^2+y^2=20\,y$, (b) $7\,x^2+5\,xy+y^2=12$,
(c) $x+y=10$, (d) $xy+20=0$,
(e) $y=k\,2^{-x^2}$, (f) $y=\dfrac{20\,x}{x^2+1}$,
(g) $x+y=xy$, (h) $x^3+y^3=1000$,
(i) $(x^2+y^2)^2=x^2-y^2$, (j) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=10^{\frac{2}{3}}$.

2. (a)–(g) Plot enough points on each curve in Ex. 1 (a)–(g), respectively, to see the locus and verify the tests made.

3. If each following curve be reflected in the line $y=x$, what will the new equation be (when solved for y)?

(a) $y=x^2$, (b) $y=x^3$, (c) $y^2=x^3$.

4. Which of the following equations are homogeneous? (Each of the letters denotes a length.)

(a) $x^2+(y-k)^2=a^2$, (b) $y^2=4\,p^2(x-h)$,
(c) $x^3+y^3-xy=0$, (d) $xy=\sqrt{x^4+y^4}$,
(e) $\sqrt{\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}}=\dfrac{1}{k}$, (f) $\left(\dfrac{x}{a}\right)^{\frac{2}{3}}+\left(\dfrac{y}{b}\right)^{\frac{2}{3}}=1$.

5. In rationalizing the equation $\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a$, a student got finally $x^2(a^2-c^2)+y^2=a^4-a^2c^2$. Is this result possible? At an intermediate stage the equation was $a\sqrt{(x+c)^2+y^2}=a^2+cx$. Was this latter homogeneous?

6. Of what "dimensions" or degree in terms of length is an *angle* θ? Its sine? Is this homogeneous: $c^2=a^2\theta^2+b^2\sin^2\theta$?

7. The formula in Ex. I, p. 13, should read: $R = (a^4y^2 + b^4x^2)^{\frac{3}{2}}/a^4b^4$. Show this to be homogeneous.

§ 7. Trigonometric Principles. In what follows you will often need to have at your finger-tips a number of facts concerning the trigonometric functions:

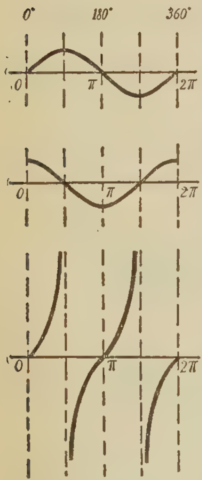


FIG. 6.

(1) *Their definitions:* in right triangles, or for large angles. (Appendix, p. 490.)

(2) *How often they repeat.* The period of the sine and cosine is 2π radians or 360°; that of the tangent is $\pi^{(r)}$ or 180°. Similarly for the reciprocal functions.

(3) *Values for the Quadrantal Angles,* 0°, 90°, 180°, etc.; or $0^{(r)}$, $\frac{\pi}{2}^{(r)}$, $\pi^{(r)}$, etc.; and conversely, for what angles the sine or cosine is 0, 1 or -1, and the tangent 0 or ∞.

(4) *Use of Tables, for Large Angles.* Combine always with 180° or 360°, taking the same function, and choosing the proper sign for the quadrant in question. (Cf. *Intro.*, §§ 260–261.)

To handle all such matters without hesitation fix the necessary *facts* in mind. To *see* what you are doing, fix indelibly in your visual memory the shape of standard graphs of the sine, cosine, and tangent.

(5) *Positive and Negative Angles.* Changing the sign of an angle has no effect on the cosine (or secant), but reverses the signs of the sine and tangent, and of their reciprocals.

(6) *Values for Special Angles.* The angles 30°, 45°, 60°, and combinations of these with 180° and 360°, occur very often. You will profit by remembering the right triangles from which the functions can be read off.

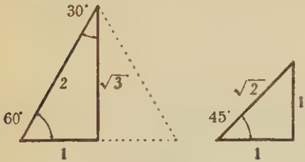


FIG. 7.

(7) *Standard Formulas:* Eight basic identities for a single angle; the Addition Formulas for two angles $(A \pm B)$; and their special form for a double angle (2θ). (Appendix, p. 490.)

EXERCISES

1. Draw from memory the graphs of $\sin\theta$, $\cos\theta$, $\tan\theta$, and then compare with Fig. 6.

2. Draw an angle of 270°, take $r=10$ and note the values of x and y for the point so obtained. Then write by inspection the values of the six trigonometric functions of 270°. Proceed likewise for 180°, 90°, and 0°. Use care as to $+$ or $-$.

3. Look up, or otherwise find, values for $\sin 320°$, $\cos(-70°)$, $\tan(-160°)$, $\text{ctn}\,225°$, $\sec 35°$, $\csc 150°$.

4. If 180° or $\pi^{(r)}$ be added to any angle, what effect will this have upon the sine? Cosine? Tangent? Secant?

5. Transform each following expression into sines and cosines, and simplify if possible:

(*a*) $\dfrac{\tan\theta}{\sec^2\theta}$, (*b*) $\dfrac{\csc^3\theta}{\text{ctn}^4\theta}$, (*c*) $\dfrac{\tan^2\theta\sec\theta}{\csc\theta}$.

6. If $x=5\cos\theta$ and $y=4\sin\theta$, show that

$$\frac{x^2}{25}+\frac{y^2}{16}=1.$$

7. If $x=8\cos^3\theta$ and $y=8\sin^3\theta$, show that $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4$.

8. Reduce $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}$ to $(\sec\theta+\tan\theta)$. [Hint: First multiply and divide, under the radical, by $(1+\sin\theta)$.]

9. Similarly make the following reductions:

(*a*) $\dfrac{1}{1-\cos\theta}$ to $\dfrac{1+\cos\theta}{\sin^2\theta}$; (*b*) $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$ to $(\csc\theta-\text{ctn}\,\theta)$.

10. If $a \cos \theta - b \sin \theta = 0$, find $\sin \theta$ and $\cos \theta$. Then show that $a \sin \theta + b \cos \theta$ would reduce to $\sqrt{a^2+b^2}$.

11. Express $\sin^{m-2} \theta \cos^2 \theta$ in terms of $\sin \theta$ only.

12. Show that $\tan^7 \theta = \tan^5 \theta \sec^2 \theta - \tan^5 \theta$; also that $\tan^7 \theta = (\tan^5 \theta - \tan^3 \theta + \tan \theta) \sec^2 \theta - \tan \theta$.

13. Simplify each of the following forms:

(a) $\dfrac{\sqrt{25+25 \tan^2 \theta}(5 \sec^2 \theta)}{625 \tan^4 \theta}$, (b) $\dfrac{(4 \sec^2\theta - 4)^{\frac{3}{2}}(2 \sec \theta \tan \theta)}{64 \sec^6 \theta}$.

14. Express in a form free from multiple or combination angles:

(a) $r = 10 \sin 2\theta$, (b) $r^2 = 100 \cos 2\theta$,

(c) $r \cos (\theta - A) = p$, (d) $\sin (A+B) - \sin (A-B)$.

15. If $\sin \dfrac{A}{2} = \dfrac{3}{5}$ and $\cos \dfrac{A}{2} = \dfrac{4}{5}$, find $\sin A$ and $\cos A$.

16. Change $40 \sin \theta \cos \theta$ to a form whose maximum value can be recognized by inspection.

17. If $\tan \dfrac{\theta}{2} = u$, find $\sin \dfrac{\theta}{2}$ and $\cos \dfrac{\theta}{2}$. Then show that $\sin \theta = \dfrac{2u}{1+u^2}$ and $\cos \theta = \dfrac{1-u^2}{1+u^2}$.

18. Reduce to a simpler form the equation:

$$(r^2 \cos^2 \theta + r^2 \sin^2 \theta)^2 = a^2(r^2 \cos^2 \theta - r^2 \sin^2 \theta).$$

19. Is the sine curve symmetrical with respect to the origin or either axis? The cosine curve?

20. Write the equation of the straight line which passes through $(2 \cos \theta, 2 \sin \theta)$ with the slope $-\text{ctn}\, \theta$. Simplify.

21. By considering 3θ as $(2\theta + \theta)$ derive a formula for $\sin 3\theta$. Express finally in terms of $\sin \theta$ alone.

22. Using the result of Ex. 21 reduce $\dfrac{\sin 3\theta}{\cos \theta}$ to $(4 \sin \theta \cos \theta - \tan \theta)$.

23. Derive a formula for $\cos 3\theta$ in terms of $\cos \theta$ alone.

PART II. POLAR COÖRDINATES

§ 8. Basic Ideas.* A point can be located by its radius vector r and its polar angle θ. These give its distance and direction from the origin, or *pole*. Negative values of r run in the opposite direction from positive values.

* Summarized from *Introduction*, Chapter X.

E.g., $(-18, 30°)$ lies 18 units from the pole in the direction opposite to the 30° direction; and thus coincides with $(18, 210°)$.

Negative radii vectores are not essential for locating points, but they make possible somewhat simpler equations for many curves.

A curve can be plotted by points from its polar equation by calculating r for various angles θ. Negative values of r may or may not give different portions of the curve than are obtained from positive values of r for some angles.

Ex. I. Plot $r = a \sin 2\theta$.

When $\theta = 15°$, $2\theta = 30°$, $\sin 2\theta = .5$; hence $r = .5\,a$. Similarly for the other values in the table:

θ	0	15	30	45	60	75	90	(deg.)
r	0	.5	.87	1	.87	.5	0	$(\times a)$

Using any convenient length for a and plotting these points, we get the loop in Quadrant I of Fig. 8. When θ runs from 90° to 180°, 2θ runs from 180° to 360° and its sine runs through negative values; 0, $-.5$, $-.87$, etc. As r is negative, the points are thrown from Quadrant II, where θ itself falls, across to Q IV. This gives the loop shown dotted.

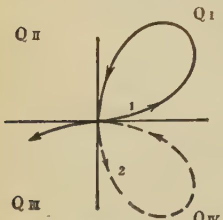


FIG. 8.

As θ goes on from 180° to 270°, r repeats the + values in the table above, giving a loop in Q III. As θ runs from 270° to 360° r is again negative and the points are thrown from Q IV across to Q II. Thus the complete curve has four loops. (Fig. 11, p. 19.)

Ex. II. Plot $$r = 10 \sec(\theta - 60°).$$

This may be written: $$r = \frac{10}{\cos(\theta - 60°)}.$$

When $\theta = 0°$, $\theta - 60° = -60°$; hence $\cos(\theta - 60°) = .5$, making $r = 20$. Similarly for the other values in the table:

θ	0	30	60	90	120	150	180	210 ⋯ 300	330	360
r	20	11.5	10	11.5	20	∞	-20	-11.5 ⋯ -20	∞	20

Plotting these points gives Fig. 9. (Points with negative values of r, between $\theta=150°$ and $\theta=330°$, are thrown across and merely repeat other points.) The locus appears to be a straight line; and indeed can be proved to be such. (Cf. p. 20, Ex. II.)

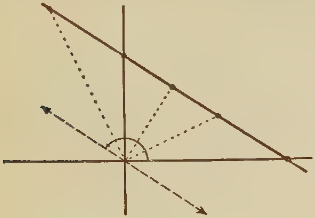


FIG. 9.

EXERCISES

1. Plot the curve in Ex. I above, from $\theta=0°$ to $\theta=360°$.

2. Plot each of the following curves from $\theta=0°$ to $\theta=360°$:

(*a*) $r=.05\,\theta$, (*b*) $r=2\cos\theta$, (*c*) $r=\cos 2\,\theta$,
(*d*) $r=20/\theta$, (*e*) $r=\sin 3\,\theta$, (*f*) $r=1+\cos\theta$.

3. Plot $r=\sec\theta$. [Cf. Ex. 2(*b*), also Ex. II, p. 17.]

4. Plot $r^2=\cos 2\,\theta$. [Cf. Ex. 2(*c*).]

§ **9. Loop Curves by Inspection.** Any curve whose equation has one of the forms

$$r=a\sin n\theta, \qquad r=a\cos n\theta,$$

can be drawn quickly by inspection.

As an angle grows larger and larger, its sine becomes zero at regular intervals and runs alternately through $+$ and $-$ values. Likewise the cosine. Wherever r becomes zero, the curve starts out from, or comes into, the origin. Successive loops, with r alternately $+$ and $-$, will swing alternately from the quadrant in which θ lies to the quadrant opposite to θ.

The standard sine and cosine graphs, p. 14, show for what angles each function is zero; also at which such angles the function is about to become $+$ rather than $-$.

Ex. I. Draw $r=10\cos 3\,\theta$.

A cosine is zero when its angle $(3\,\theta)$ is $-90°$, $90°$, $270°$, $450°$, $\cdots$.

$$\therefore\ r=0 \text{ when } \theta=-30°,\ 30°,\ 90°,\ 150°,\ \cdots.$$

We mark these angles, every 60°, where loops begin or end, noting that from $-30°$ to $30°$, r is $+$. Hence successive intervals run thus:

$-30°$ to $30°$, $r+$, loop 1 *with* θ,
$30°$ to $90°$, $r-$, loop 2 opposite θ,
$90°$ to $150°$, $r+$, loop 3 *with* θ,
$150°$ to $210°$, $r-$, loop 4 opposite θ.

Loop 4 repeats loop 1; 5 repeats 2; 6 repeats 3. The complete curve consists of three loops, each traced twice while θ runs from 0° to 360°. (Fig. 10.)

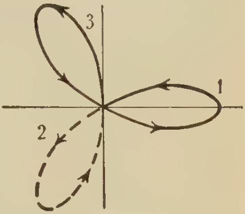


Fig. 10.

The maximum numerical value of r is evidently 10 for each loop, and comes at the middle of the loop.

Ex. II. Draw $r = a \sin 2\,\theta$, taking a positive.

The sine is zero when the angle (here $2\,\theta$) is 0°, 180°, 360°, ⋯.

Hence

$$r = 0 \text{ at } \theta = 0°,\ 90°,\ 180°,\ \cdots,$$

and r is $+$ from $\theta = 0°$ to $\theta = 90°$. Evidently then the loops begin every 90° and run as in Fig. 11.


Fig. 11.

Here r is $-$ in loops 2 and 4, shown dotted. A point traversing the four loops in order makes *no sudden change or reversal of direction* anywhere. (Cf. Fig. 8, p. 17.)

EXERCISES

1. Draw the following curves by inspection from $\theta = 0°$ to $\theta = 360°$, using a dotted line for loops in which r is negative. Mark the values of θ at which loops begin and end; and number the loops in the order traversed. Show the maximum r in each exercise.

(*a*) $r=\sin\theta$, (*b*) $r=5\sin 3\theta$, (*c*) $r=10\sin 5\theta$,
(*d*) $r=\cos\theta$, (*e*) $r=2\cos 3\theta$, (*f*) $r=2\cos 5\theta$,
(*g*) $r=\sin 2\theta$, (*h*) $r=10\cos 2\theta$, (*i*) $r=\sin 4\theta$,
(*j*) $r=2\cos 6\theta$, (*k*) $r=\cos\left(\frac{\theta}{2}\right)$, (*l*) $r=\sin\left(\frac{3\theta}{2}\right)$.

2. Do the results of Ex. 1 suggest any rule as to the number of loops in the case of $n\theta$ if n is even? If n is odd?

§ 10. Transformation to Rectangular Coördinates. Some curves are more easily recognized if we change from their polar to their rectangular equations. If the origin be at the pole and the positive X-axis coincide with the polar axis, we have the relations:

$$\sin\theta=\frac{y}{r},\qquad \cos\theta=\frac{x}{r},\qquad r=\sqrt{x^2+y^2}. \tag{12}$$

We first eliminate all functions of θ, and then substitute for r. If any multiple or combination angles, such as 2θ or $\theta+30°$, are present, their functions must first be reduced to sines and cosines of θ alone.

Ex. I. $$r=10\sin\theta.$$

Using (12), this becomes

$$r=10(y/r),$$

or $$r^2=10y,$$

whence $$x^2+y^2=10y.$$

Completing the square shows a circle, center (0, 5), radius 5. (Cf. *Intro.*, § 268, Ex. I and Fig. 127.)

Ex. II. $$r=10\sec(\theta-60°).$$

Here $$r=\frac{10}{\cos(\theta-60°)}.$$

Multiplying through by $\cos(\theta-60°)$ and expanding:

$$r(\cos\theta\cos 60°+\sin\theta\sin 60°)=10.$$

Substituting for $\cos\theta$ and $\sin\theta$ by (12) and multiplying out:

$$x \cos 60° + y \sin 60° = 10.$$

This, being a linear equation, represents a straight line.

The line has already been drawn in Fig. 9, p. 18.

§ 11. **From Rectangular to Polar.** Sometimes the polar equation of a curve is the simpler. To obtain it from the rectangular equation, when the pole is taken at the origin and the polar axis along the positive end of the X-axis, use:

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{13}$$

Ex. I. $\quad (x^2+y^2)^2 = 100(x^2-y^2).$

By (13): $\quad r^4 = 100\, r^2(\cos^2\theta - \sin^2\theta). \qquad (14)$

One solution of (14) is:

$$r^2 = 100(\cos^2\theta - \sin^2\theta).$$

By a double-angle formula this becomes

$$r^2 = 100 \cos 2\,\theta.$$

Remarks. (I) Plotting by points we get the curve in Fig. 12. This is called a *Lemniscate*. (From $\theta = 45°$ to $\theta = 135°$, r is imaginary; also from 225° to 315°.)

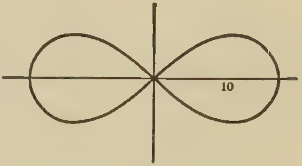


FIG. 12.

(II) Another solution of (14) is:

$$r^2 = 0 \qquad \text{regardless of } \theta,$$

which indicates that every line through the origin cuts the curve at that point, irrespective of its direction, and cuts it *doubly* there.

EXERCISES

1. Change to the polar equation, solved for r. If you recognize any curve by either form of equation, state what it is.

(*a*) $x = 4$, (*b*) $y + 8 = 0$,
(*c*) $xy = 20$, (*d*) $x^2 + y^2 - 6\,y = 0$,
(*e*) $(x^2+y^2)^3 = (x^2-y^2)^2$, (*f*) $x^3 + y^3 - 6\,xy = 0$,
(*g*) $(x^2+y^2-ax)^2 = b^2(x^2+y^2)$, (*h*) $(x^2+y^2)(x-a)^2 = b^2x^2$.

2. Change to the rectangular equation, free from radicals. If you recognize any curve, state what it is.

(a) $r=5\sin\theta$, (b) $r=10\csc\theta$,
(c) $r=10\tan\theta\sec\theta$, (d) $r=5\sin 2\theta$,
(e) $r=2(1-\cos\theta)$, (f) $r=2a(\sec\theta-\cos\theta)$,
(g) $r^2=\dfrac{36}{4\cos^2\theta+9\sin^2\theta}$, (h) $r=\dfrac{12}{\sin(\theta-30°)}$,
(i) $r\cos(\theta+45°)=10$, (j) $r^2-8r\cos(\theta-30°)+7=0$.

3. What is the rectangular equation of a horizontal line 7 units above the X-axis? Hence what polar equation?

4. Like Ex. 3 for a vertical line 20 units to the left of the Y-axis.

5. What is the rectangular equation of a circle with center (4, 3) and radius 5? Hence what polar equation?

6. Find directly or indirectly the polar equation of the parabola with vertex (0, 0°) and focus (5, 90°).

§ 12. Polar Equation of a Straight Line. For a straight line which passes through the pole or origin, the equation is

$$\theta=k, \text{ some constant.}$$

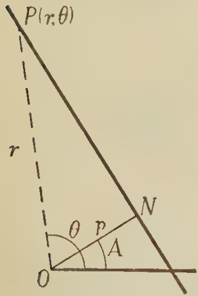


Fig. 13.

Any other line is at some distance p from the origin, and the "normal" or perpendicular ON has some polar angle A. (Fig. 13.) Then from $\triangle ONP$:

$$r\cos(\theta-A)=p. \tag{15}$$

This equation holds for any point P of the line, and no others. (Cf. Ex. II, p. 17, and Fig. 9.)

By comparison with (15) we can recognize various equations as representing straight lines. *E.g.*,

$$r\cos(\theta+30°)=12$$

comes under (15) with $p=12$ and $A=-30°$. The normal is 12 units long, in the direction $-30°$. The line itself has, therefore, an inclination of 60°.

Some equations must first be reduced to the form (15).

Ex. I. Identify $r \cos (\theta - 10°) = -20.$
Changing the sign of r moves any point to the opposite position through the origin. Hence the present line lies opposite to the position it would have if the right member were $+20$. The normal, of length 20, runs into the third quadrant with $A = 190°$.

Ex. II. Identify $r \sin \theta = 20.$ (16)
The sine of any angle equals the cosine of an angle 90° smaller.*
Hence equation (16) may be written:

$$r \cos (\theta - 90°) = 20.$$

The normal runs in the 90° direction, and the line is horizontal.

Check. Transforming (16) to rectangular coördinates gives $y = 20$.

§ 13. Polar Equation of a Circle. In problems involving circles we can usually choose the origin or pole in some special position relative to a circle, as in (I) or (II) below, and thereby obtain a very simple equation for the circle. But occasionally we have to take the pole in a general position, as in (III). In any case, let a denote the radius and (r, θ) any point on the circle.

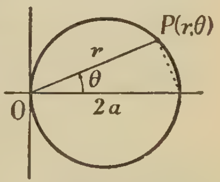


FIG. 14.

(I) *Pole at the Center.* Here, for every θ, $r = a$ (or else $-a$, using the reverse direction). The equation is simply

$$r = \pm a, \qquad \text{or} \qquad r^2 = a^2, \text{ combined.}$$

(II) *Pole on the Circle; Axis Passing through the Center.* Here, it is seen from Fig. 14 that, for every point on the circle and for no others:

$$r = 2\,a \cos \theta. \tag{17}$$

This is the form of equation which we shall most frequently need.

(III) *Pole anywhere;* or general position of a circle with respect to the origin and polar axis. Let the center of the circle have the coördinates (c, A). Then, from Fig. 15, by the Law of Cosines:

$$a^2 = r^2 + c^2 - 2\,rc \cos (\theta - A),$$

or

$$r^2 - 2\,rc \cos(\theta - A) + (c^2 - a^2) = 0. \tag{18}$$

* Cf. Appendix, p. 490, or *Intro.*, p. 357, (8).

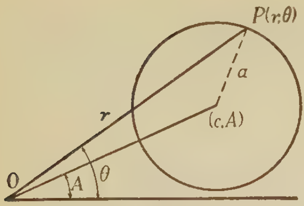


FIG. 15.

For any specified value of θ within certain limits, (18) gives two values of r. These belong to the two points on the circle in the specified direction from O.

The preceding special equations of a circle are also obtainable directly from (18) by inserting special values for c and A. (See Ex. 8 below.)

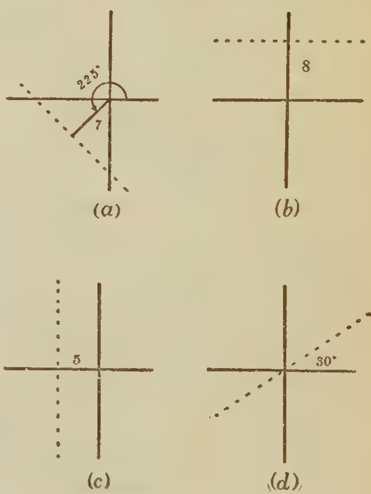
Equations (17) and (18) need not be memorized outright. But we should so thoroughly understand their derivation as to be able to obtain the equations immediately from a roughly drawn figure.

Ex. I. Will a line drawn from the pole in the direction $\theta = 80°$ meet the circle with center (25, 20°) and radius 21?

By (18) the equation of the circle is

$$r^2 - 50\, r \cos (\theta - 20°) + 184 = 0.$$

When $\theta = 80°$, $\cos (\theta - 20°) = \cos 60° = \frac{1}{2}$; and $r^2 - 25\, r + 184 = 0$. Here $b^2 - 4\, ac = 625 - 736$. As this is negative, the values of r are imaginary. The line does not meet the circle. (Verify by drawing.)



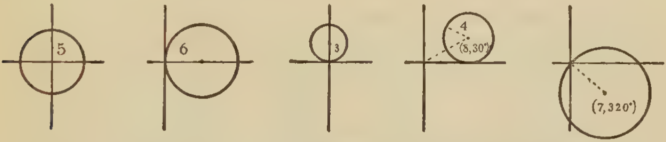
EXERCISES

1. For each dotted straight line in the following figure read off the values of A and p; and write the equation of the line.

2. In Ex. 1 (b)–(d) write the rectangular equation by inspection of the figure, change to polar form, and check.

3. Get the rectangular equations of the lines in Ex. 1 (a), (b), by transforming the polar equations found above.

4. For each circle in the following figure note the center and radius, and write the polar equation directly.



(*a*) (*b*) (*c*) (*d*) (*e*)

5. (*a*)–(*c*). In Ex. 4 (*a*)–(*c*) write the rectangular equation by inspection, change to polar form, and check.

6. From the polar equations in Ex. 4 (*d*), (*e*), get the rectangular equations. Check by inspection of the figure.

7. A circle of radius 5 has its center at (10, 10) in rectangular coördinates. Find the polar equation of the circle in two different ways.

8. Show that equation (18) reduces to (17) if we put $c=a$ and $A=0$. (What significance has the extra factor, $r=0$?) Also obtain the equation $r^2=a^2$ in case (I) by substituting a special value for c in (18).

9. (*a*) By drawing a figure and using trigonometry, find the polar equation of a circle of radius 10, with the lowest point L taken as pole and the tangent at L as the polar axis. (*b*) Obtain the same equation by giving special values to c and A in (18).

10. Does the circle in Ex. I, p. 24, cut the radius vector for which $\theta=70°$? If so, what are the two values of r?

PART III. PARAMETRIC EQUATIONS

§ 14. Auxiliary Variables. In some cases a curve is best studied as the path of a moving point. The position of the point at any instant may be described by equations which give its coördinates (x, y) or (r, θ) in terms of the time t. Or, without using the idea of time, the position along the curve may be described in terms of some convenient angle, distance, or other auxiliary variable.

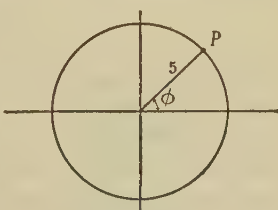


FIG. 16.

E.g., in Fig. 16, we have at each point P of the circle:

$$x=5\cos\phi, \qquad y=5\sin\phi, \tag{19}$$

the auxiliary angle being denoted by ϕ (Greek letter "phi").

For each value of ϕ there is a single point; and as ϕ varies, P travels around the circle.

§ 15. Parametric Equations. Such auxiliary variables as t and ϕ above are called *parameters*. Equations like (19), which express in terms of a parameter the coördinates of any point on a curve, are called *parametric equations* of the curve. A single curve may have many different parametric equations, according to the choice of the auxiliary variable. But eliminating the parameter should always give the same rectangular (or polar) equation of the curve.

If trigonometric functions are involved, a common method of elimination is to solve the parametric equations for some two functions in terms of x and y (or r and θ), and then substitute in one of the "identities." (Appendix, p. 490.)

In Fig. 16 the parameter ϕ happens to be the same as the polar angle θ. But this is usually not the case when using an auxiliary angle as the parameter.

Ex. I. Find what curve has the parametric equations $x=5\sec\phi$, $y=3\tan\phi$.

Recalling that for any angle whatever $\sec^2\phi=1+\tan^2\phi$, we see from the given equations that

$$\left(\frac{x}{5}\right)^2=1+\left(\frac{y}{3}\right)^2,$$

or

$$\frac{x^2}{25}-\frac{y^2}{9}=1. \tag{20}$$

This is the equation of an hyperbola.

§ 16. The Ellipse. Some problems concerning ellipses are most easily solved by using parametric equations. The parameter most frequently used is the "eccentric angle," $\angle AOQ$, running not to the point P itself but to the corre-

sponding point Q on the major circle, where the circle meets the vertical line through P. At Q:

$$X = a \cos \phi, \qquad Y = a \sin \phi.$$

By § 214, *Intro.*, the ordinate of P is $y = (b/a)\,Y$. Hence for P:

$$\boldsymbol{x = a \cos \phi, \quad y = b \sin \phi.} \tag{21}$$

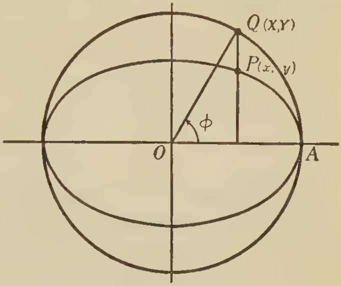


FIG. 17.

These equations are important in Astronomy and elsewhere. That they represent an ellipse can be seen directly by eliminating ϕ, which gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{(Why?)}$$

§ **17. Changing to Parametric Equations.** Sometimes it is helpful to change from the rectangular equation of a curve to parametric equations without regard to the geometrical meaning of the parameter introduced. To do this, we may simply set x equal to any function of t or ϕ, at pleasure, and find from the given equation what value y must then have.

To illustrate, let us find two different sets of parametric equations for the rectangular hyperbola $xy = 100$:

(A) If we let $x = 10 \sin \phi$, then $y = 100/x = 10 \csc \phi$. Thus, possible equations are

$$x = 10 \sin \phi, \qquad y = 10 \csc \phi.$$

(B) If we let $y = lx$, then $lx^2 = 100$, and

$$x = \pm \frac{10}{\sqrt{l}}, \qquad y = \pm 10\sqrt{l}.$$

This last amounts to finding the intersections of the hyperbola with any straight line $y = lx$ through the origin.

A procedure which is helpful with a number of important curves will be used now to find parametric equations for

$$(2x)^{\frac{2}{3}}+(5y)^{\frac{2}{3}}=20^{\frac{2}{3}}. \tag{22}$$

Dividing through by $20^{\frac{2}{3}}$ gives

$$\left(\frac{x}{10}\right)^{\frac{2}{3}}+\left(\frac{y}{4}\right)^{\frac{2}{3}}=1.$$

We have here two quantities whose sum is 1. Hence we may put them equal to (cosine)^2 and (sine)^2 for some angle.

Let $\left(\frac{x}{10}\right)^{\frac{2}{3}}=\cos^2\phi,\qquad \left(\frac{y}{4}\right)^{\frac{2}{3}}=\sin^2\phi.$

$$\therefore\ \frac{x}{10}=\cos^3\phi;\qquad \frac{y}{4}=\sin^3\phi,$$

$$\therefore\qquad x=10\cos^3\phi,\qquad y=4\sin^3\phi. \tag{23}$$

To plot the curve we could easily calculate many points from (23). But (22) would be much more awkward.

EXERCISES

1. Plot $x=20\cos\phi$, $y=10\sin\phi$. [Take ϕ at intervals of 15° from 0° to 90°, using only three-place values. Draw the rest of the curve by inspection.] Eliminate ϕ and get the rectangular equation of the curve.

2. The same as Ex. 1 for $x=20\cos^3\phi$, $y=10\sin^3\phi$.

3. Plot $x=3t^2$, $y=t^3$, from $t=-4$ to $t=4$. Also find the rectangular equation.

4. A railway easement curve, used to join a straight track to a uniform (circular) curve, has the following equations when designed for a speed of about 33 mi./hr.:

$$x=600\left[u-\frac{u^5}{10}+\cdots\right],\qquad y=200\left[u^3-\frac{u^7}{14}+\cdots\right].$$

Plot, taking u at intervals of .1 from 0 to .5. (Here x and y are in feet.)

5. A common cycloid has the equations:

$$x=10(\phi-\sin\phi),\qquad y=10(1-\cos\phi),$$

where ϕ is the number of *radians* in an angle. Plot, taking ϕ at intervals of $\frac{\pi}{3}$ from 0 to 2π. [*E.g.*, when $\phi=\frac{\pi}{3}=1.047$, $x=1.81$; etc.]

6. The same as Ex. 5 for the involute of a circle:

$$x = 10(\cos\phi + \phi\sin\phi), \qquad y = 10(\sin\phi - \phi\cos\phi).$$

7. Find the rectangular equation for the curve $x = 3\cos\phi$, $y = 3\sin\phi$. Draw by inspection, and show the meaning of ϕ.

8. The same as Ex. 7 for the curve $x = 5\cos\phi$, $y = 3\sin\phi$.

9. The path of a ball thrown in a certain way has the equations: $x = 40\,t$, $y = 80\,t - 16\,t^2$. Find the rectangular equation, simplify it, and draw the path by inspection.

10. A pen moving vertically with a simple harmonic motion, $y = 4\sin 2\,t$ (where the units are 1 cm., 1 sec., and 1 radian), traces a curve on a strip of paper which is being pulled past horizontally with a constant speed of 1 cm./sec. What is the rectangular equation of the curve? Draw by inspection.

11. The position of a planet or periodic comet in its orbit at any time is found by the equations:

$$r = a(1 - e\cos E), \qquad \tan\theta = \frac{\sqrt{1-e^2}\sin E}{\cos E - e},$$

where a, e, are constants, and E is found from an equation involving t. For Halley's comet $a = 18$, $e = .967$, approx.: find r and θ when $E = 0°$, $90°$, $180°$.

12. Plot $x^3 + y^3 = -6\,xy$ by finding its intersections with a straight line through the origin, $y = lx$, for various values of l. Also write the parametric equations in terms of l.

13. Write parametric equations for a circle of radius 20, and show by a figure the meaning of the parameter used.

14. The same as Ex. 13 for an ellipse whose semi-axes are 15 and 5. Check by getting the rectangular equation.

§ **18. Roulettes.** When a moving curve M rolls, without slipping, along a fixed curve F in a plane, any point P upon M or rigidly connected with it traces out some curve called a *roulette*.

The shape of a roulette depends upon the nature of M and F and the position of P with respect to M. Several types will now be considered. Parametric equations will be the most convenient.

§ **19. The Cycloids.** A roulette obtained by rolling a circle along a straight line is called a *cycloid*, — a "prolate"

cycloid if P is inside of M, "curtate" if outside, and "common" if P is on M. (Cf. *Intro.*, § 276.)

Let ϕ denote the angle in radians through which the circle has rolled, a the radius, and b the distance of P from the center. Choose as origin the point on F where the circle began to roll. Then, by the right triangle method formerly used for the common cycloid (*Intro.*, p. 376), we obtain the following parametric equations:

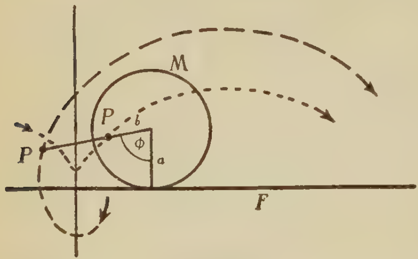


FIG. 18.

$$x = a\,\phi - b\sin\phi; \qquad y = a - b\cos\phi. \tag{24}$$

If P is on the circle, $b = a$, and equations (24) reduce to the former equations of the common cycloid:

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi). \tag{25}$$

§ 20. The Epicycloids. If one circle rolls upon another externally, any point P on the rolling circle M generates a roulette called an *epicycloid*. The shape depends on the ratio of the radii of the two circles. Evidently P strikes the fixed circle F periodically, at points whose distance apart along F equals the circumference of M. At each such point the epicycloid is found to have a sharp point or "cusp."

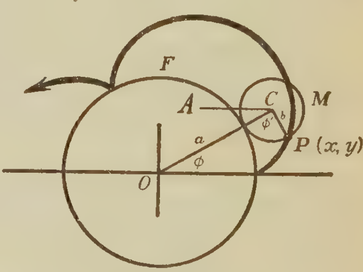


FIG. 19.

If the radius of M is precisely one-third that of F, the epicycloid will have three cusps and finish exactly three arches in one revolution. Similarly for other ratios. (Fig. 19.)

Let a and b be the radii of F and M, respectively. Then the rectangular coördinates of the moving center C, with axes as shown, are:

$$X=(a+b)\cos\phi, \qquad Y=(a+b)\sin\phi. \tag{26}$$

For P these must be decreased by $b\cos(\angle ACP)$ and $b\sin(\angle ACP)$. But $\angle ACP=\phi'+\phi$.
Hence

$$\begin{aligned} x&=(a+b)\cos\phi-b\cos(\phi'+\phi),\\ y&=(a+b)\sin\phi-b\sin(\phi'+\phi). \end{aligned} \tag{27}$$

The arc of M which has rolled on F equals $b\phi'$; the arc of F on which it has rolled equals $a\phi$. Since the two arcs must be equal,

$$b\phi'=a\phi.$$

Thus ϕ' can be replaced in (27) by $a\phi/b$.

To get more convenient equations, let $a=nb$, where n may or may not be an integer. Then $\phi'=n\phi$, and (27) become

$$\begin{aligned} \boldsymbol{x}&=\boldsymbol{b[(n+1)\cos\phi-\cos\overline{n+1}\,\phi]},\\ \boldsymbol{y}&=\boldsymbol{b[(n+1)\sin\phi-\sin\overline{n+1}\,\phi]}. \end{aligned} \tag{28}$$

These equations are for an epicycloid of n cusps, if n is an integer. The minimum distance from the origin is nb, and the maximum is $(n+2)b$.

Remarks. (I) Equations (28) need not be memorized. But try to fix their general appearance in mind, so that when equations of the sort are encountered you will think of these, and turn back to (28) to check the resemblance.

(II) Ancient astronomers tried to describe the motions of the planets by a system of epicycles, — circle on circle on circle, etc. Bizarre as this idea is, it gives a rough picture of the geometrical significance of certain trigonometric series used in the modern study of the motions. For another application of epicycloids, see § 294.

(III) If n is a rational number, the epicycloid is a closed curve; if n is irrational, P never returns to the starting point.

EXERCISES

In these exercises let the rolling be without slipping.

1. A circle of radius 8 rolls along a straight line. Draw roughly the cycloids generated by points on a radial line at distances of 2, 8, and 10 from the center. Write parametric equations for each.

2. The same as Ex. 1 for a circle of radius 5, and points at distances of 1, 5, and 6 from the center.

3. Draw roughly the epicycloid generated by a point on a circle of radius 3 which rolls upon a circle of radius 12. Write parametric equations for the epicycloid.

4. The same as Ex. 3 if the rolling circle has:

(*a*) radius 4, (*b*) radius 6, (*c*) radius 12.

5. Draw Fig. 19 for the case where $n=3$ and obtain equations corresponding to (26) to (28) inclusive.

6. What is the meaning of b in equations (28)? Of n? Draw the following roughly by recognizing the equations:

(*a*) $x=4(6\cos\phi-\cos 6\phi)$, $y=4(6\sin\phi-\sin 6\phi)$;
(*b*) $x=2\cos\phi-\cos 2\phi$, $y=2\sin\phi-\sin 2\phi$,
(*c*) $x=2(\phi-\sin\phi)$, $y=2(1-\cos\phi)$.

7. A circle rolls on the convex side of the parabola $y=x^2$. Show the general appearance of the roulette generated by that point on the circle which comes into contact with the parabola at the vertex. [Draw the parabola lightly and the roulette heavily to display the latter.]

8. A long straight line rolls around a circle. Show the appearance of the roulette traced by any point on the line after it has been in contact with the circle.

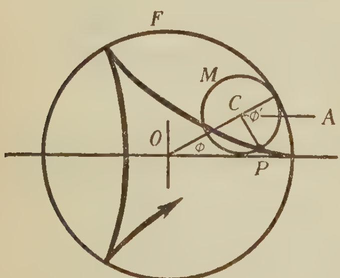


Fig. 20.

§ 21. The Hypocycloids. If one circle M rolls upon another circle F *internally*, any point P on M generates a roulette called a *hypocycloid*. Its shape and number of cusps depend on the ratio n of the radii a and b of circles F and M. (Fig. 20.)

The coördinates of C are $X=(a-b)\cos\phi$, $Y=(a-b)\sin\phi$. For P, X must be increased by $b\cos(\phi'-\phi)$ and Y decreased by $b\sin(\phi'-\phi)$.
Hence

$$\begin{aligned} x&=(a-b)\cos\phi+b\cos(\phi'-\phi),\\ y&=(a-b)\sin\phi-b\sin(\phi'-\phi). \end{aligned} \tag{29}$$

As for the epicycloid, $a\phi=b\phi'$, making $\phi'=a\phi/b=n\phi$. Thus

$$\begin{aligned} x&=b[(n-1)\cos\phi+\cos\overline{n-1}\,\phi],\\ y&=b[(n-1)\sin\phi-\sin\overline{n-1}\,\phi]. \end{aligned} \tag{30}$$

These equations are for a hypocycloid of n cusps, if n is an integer. The minimum distance from the origin is $(n-2)b$ and the maximum nb.

§ 22. **Special Cases.** If $n=2$, equations (30) reduce greatly, and show that a hypocycloid of two cusps is *a diameter of the fixed circle.* (Ex. 6, p. 34.)

Again, if $n=4$, equations (30) become

$$\begin{aligned} x&=b(3\cos\phi+\cos 3\phi),\\ y&=b(3\sin\phi-\sin 3\phi). \end{aligned} \tag{31}$$

But, by Ex. 21, 23, p. 16,

$$\begin{aligned} \cos 3\phi&=4\cos^3\phi-3\cos\phi,\\ \sin 3\phi&=3\sin\phi-4\sin^3\phi. \end{aligned} \tag{32}$$

Substituting these values in (31) and replacing $4b$ by a, we have

$$x=a\cos^3\phi, \qquad y=a\sin^3\phi. \tag{33}$$

These simplified parametric equations for the hypocycloid of four cusps are considerably used; also the rectangular equation, obtained by eliminating ϕ:

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}. \tag{34}$$

(See Fig. 26 C, p. 62.)

EXERCISES

1. A circle of radius 3 rolls on the inside of a circle of radius 12. Draw roughly the hypocycloid generated by any point on the rolling circle. Write parametric equations for the hypocycloid. (Also write an alternative form in this case.)

2. The same as Ex. 1 (without the last step) if the rolling circle has:

(*a*) radius 4; (*b*) radius 6; (*c*) radius 12.

3. Draw Fig. 20 for the case where $n=3$, and derive the equations corresponding to (30).

4. Verify the reduction of (31) to (33) by use of (32).

5. Derive (34) from (33). [Solve for $\sin\phi$ and $\cos\phi$, and combine.]

6. Show that in a hypocycloid of two cusps the motion is simple harmonic if the moving circle rolls with constant speed. [Cf. *Intro.*, § 277, (15).]

7. Recognize and draw:

(*a*) $x=5(2\cos\phi+\cos 2\phi)$, $y=5(2\sin\phi-\sin 2\phi)$;

(*b*) $x=7(\phi-\sin\phi)$, $y=7(1-\cos\phi)$.

§ **23. Remarks on Chapter I.** The foregoing equations and methods have been covered not as an end in themselves but as a means to an end. The object is a ready facility in dealing with the curves which will be encountered in our chief applications of calculus to scientific fields. Further methods will be shown later, as needed.

We should now know how to plot a curve from its various equations by finding points, or recognize it by inspection if an elementary curve; how to find its intercepts, and its intersections with other curves; how to recognize symmetry with respect to the origin, an axis, or the 45° line; how to change from rectangular to polar form, or vice versa, or from either of these to parametric equations, or the reverse; and how to check by the homogeneity test. We should also be able to write quickly the equations of the elementary curves in various given positions; and have a ready command of the trigonometric information listed in § 7.

If doubtful as to any of these matters, you had best review and acquire a sure grasp before proceeding. Much time and trouble will be saved later by doing this now.

EXERCISES FOR REVIEW

1. Find the equation of a straight line which cuts the parabola $y=x^2$ at points where $x=-2$ and $x=-4$.

2. Find the polar equation of a circle of radius 6, taking the pole on the curve and the axis as simply as possible.

3. In Ex. 2 transform to rectangular coördinates and check.

4. Find in two ways the polar equation of a vertical straight line a units to the left of the pole. Also show that your equation is homogeneous.

5. Find the equation of a straight line through the midpoint of (5, 2) and (11, -8) perpendicular to the line $3x-2y=7$.

6. Write the rectangular equation for a general hypocycloid of four cusps if the X- and Y-axes pass through the cusps. Show it homogeneous. Also show that it meets the tests for four kinds of symmetry. Write the equation for some other well-known curve which meets all these tests.

7. Test the homogeneity of equations (7), (9), (13), (15), (17), (18), (21), (24), (25), (28), (30).

8. Write parametric equations for an ellipse whose longest and shortest diameters are 14 and 8. Check by eliminating the parameter. Also show the meaning of the parameter by a drawing.

9. Simplify $\dfrac{\sin 2\theta}{\cos\theta}$ and $\dfrac{1-\cos 2\theta}{\sin\theta}$.

10. Suggest two methods of calculating numerous points on the curve $x^5+y^5=x^3y^3$ more conveniently than by directly substituting values for x.

11. Reduce to a standard form and draw roughly:

(*a*) $x^2+y^2=20y$, (*b*) $x^2-30x+9y=0$,

(*c*) $4x^2+y^2=20x$, (*d*) $9x^2-4y^2-32y=28$.

12. Plot by points: $2x^2+2xy+y^2=9$.

[First solve for y and note that the curve is real between, or else beyond, certain values of x.]

13. The same as Ex. 12 for each of the following:

(*a*) $x^2+4xy+2y^2+2x+3=0$; (*b*) $5x^2+4xy+y^2-6x-2y-7=0$.

CHAPTER II

DERIVATIVES

PART I. FAMILIAR PRINCIPLES *

§ **24. Definition.** Let y be a function of x, and Δy the increment or change in y produced by a change Δx in x. If $\Delta x \to 0$, the fraction $\Delta y/\Delta x$ usually approaches a limit; and that limit is called *the derivative of y with respect to x*, written dy/dx. That is,

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{\mathrm{L}} \left(\frac{\Delta y}{\Delta x}\right). \tag{1}$$

Ex. I. If $\qquad y = x^2 - 10\,x,$

then, after x and y have increased:

$$y + \Delta y = (x + \Delta x)^2 - 10(x + \Delta x).$$

Multiplying out and subtracting the original terms:

$$\Delta y = 2\,x\Delta x - 10\,\Delta x + \Delta x^2,$$

$$\frac{\Delta y}{\Delta x} = 2\,x - 10 + \Delta x.$$

The limit of this as $\Delta x \to 0$ is

$$\frac{dy}{dx} = 2\,x - 10. \tag{2}$$

The operation of finding dy/dx in any way is called *differentiation*. The method just illustrated is called the Δ-process or increment method.

When a function is written symbolically $f(x)$, its derivative is usually written simply $f'(x)$. Also $\dfrac{d}{dx}$ placed before a quantity denotes " the derivative with respect to x of " that quantity.

* Cf. *Introduction*, Chapters III, VII, X.

§ 25. **Interpretations.** The fraction $\Delta y/\Delta x$ is the average rate at which y increases per unit change in x during the interval Δx. Hence dy/dx is the *instantaneous rate* at the point where the interval began. This rate may be various things, according to the meaning of y and x.

If y is the ordinate of any point on a curve, and x the abscissa, then dy/dx is the slope of the curve at that point. If y is the distance traveled by an object, and x the time elapsed, then dy/dx is the speed at the instant considered. And so on. (Cf. § 27.)

The rate at which any quantity is changing is represented on some scale by the slope of the graph of the quantity. The rate or slope is also called the *gradient* at the point in question.

Remark. Either from the limit definition or from the rate interpretation, it follows that dx/dy is the reciprocal of dy/dx, when both exist.

§ 26. **Basic Tests.** To ascertain whether y is instantaneously *increasing* or *decreasing*, while x increases, see whether dy/dx is $+$ or $-$ at the point in question.

If dy/dx is zero or infinite at the point, test its sign just before and just after. If the sign changes from $+$ to $-$, then y has a *maximum* value at the point. (That is, it is a maximum for the immediate vicinity. There may be larger values elsewhere.) Similarly, where dy/dx changes from $-$ to $+$, y has a *minimum*.

If $y=x^2-10\,x$ (as in Ex. I, § 24), then $dy/dx=2\,x-10$. This is zero at $x=5$, and nowhere else. At $x=4$, dy/dx is $-$, and y is decreasing. At $x=6$, dy/dx is $+$, and y is increasing. Thus y reaches a minimum at $x=5$, that minimum being $y=-25$.

§ 27. **Higher Derivatives.** The derivative of dy/dx is called the second derivative of y, written d^2y/dx^2, or $f''(x)$. Similarly for d^3y/dx^3, and so on.

To find the rate at which any quantity is changing, we must find the derivative of *that* quantity, no matter how many differentiations may already have been performed.

At any point (x, y) of a curve, dy/dx is the *slope*, or rate at which the height y is changing per horizontal unit; d^2y/dx^2 is the *flexion*, or rate at which the slope is changing; d^3y/dx^3 is the rate at which the flexion is changing.

§ 28. Differentiation Formulas. The derivatives of many functions can be written at sight by means of the formulas below, which were derived in the *Introduction*.

We should memorize not the formulas themselves but verbal statements of the *principles* which the formulas symbolize. *E.g.*, the derivative of a constant is zero; the derivative of a constant times a variable quantity is equal to the constant multiplier times the derivative of the variable quantity; and so on.

$$\frac{d}{dx}(c) = 0. \tag{3}$$

$$\frac{d}{dx}(ku) = k\frac{d}{dx}(u). \tag{4}$$

$$\frac{d}{dx}(u+v+w) = \frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}. \tag{5}$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}. \tag{6}$$

$$\frac{d}{dx}(\log u) = \frac{\log e}{u}\frac{du}{dx}. \tag{7}$$

For the base e this reduces to $\frac{1}{u}\frac{du}{dx}$; for the base 10, to $\frac{M}{u}\frac{du}{dx}$, M being .43429, approx.

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}. \tag{8}$$

$$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}. \tag{9}$$

$$\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}. \tag{10}$$

If the angle is in degrees, multiply these derivatives by $\frac{\pi}{180}$, $=$ .017453, approx.

$$\frac{d}{dx}(uv) \qquad = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{11}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) \qquad = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}. \tag{12}$$

EXERCISES

All the letters denote variables, except a, b, c, e, g, k, m, n, and π.

1. Write dy/dx by inspection if $y = x^3 + 4x^2 - 7x + 11$. Check by using the Δ-process.

2. Differentiate:

(*a*) $(x^2+4x+2)^8$, (*b*) $(1+\sin\theta)^3$, (*c*) $\cos^4\theta$,

(*d*) $\log(x^4-1)$, (*e*) $5\sin(10t+.3)$, (*f*) $.7e^{-5x^2}$.

3. If $y = \frac{11}{7x^2}$, show by the Δ-process that $\frac{dy}{dx} = -\frac{22}{7x^3}$. Check by using (6) for a negative power. Note the steps in the reduction. Practice writing the derivatives of the following immediately in reduced form:

$$\frac{5}{2x^8}, \quad \frac{a}{3x^4}, \quad \frac{b}{cx^6}, \quad \frac{k}{\pi x^2}, \quad \frac{1}{gx}.$$

4. Differentiate with respect to the variable involved, and express each derivative in the most conveniently usable form:

(*a*) $y = 6x^{\frac{3}{2}}$, (*b*) $z = \pi\sqrt{t^5}$, (*c*) $v = 8\sqrt{s}$,

(*d*) $u = k\sqrt[3]{r^4}$, (*e*) $S = \sqrt[3]{V^2}$, (*f*) $r = 2.387\sqrt[3]{V}$.

5. Find the second derivative of each of the following:

(*a*) $Q = 1000\,e^{.08\,T}$, (*b*) $P = \sin 5\theta + b\cos 5\theta$,

(*c*) $x = kt + \sqrt{t} + 40$, (*d*) $F = \frac{16\,000\,000}{x^2}$,

(*e*) $S = 2\pi r^2 + \frac{4k}{r}$, (*f*) $V = \log r + 8e^{-\frac{r}{2}}$.

6. How fast does $\log x$ change with x, per unit, at $x=4$? Likewise $\log_{10}x$?

7. How fast does $\cos\theta$ change with θ, per radian, at $\theta = .3^{(r)}$? The same, per degree, at $\theta = 25°$? Does it follow from (10) that the cosine always decreases as the angle increases? Explain.

8. Plot $y=.1\,x^2$, using the same scale horizontally and vertically, and taking $x=0$, 1, 2, etc., to 8. Measure the slope and inclination at $x=5$, and check by calculation.

9. The height of an arched ceiling of a hall above the floor x ft. from the center line is $h=40-.02\,x^2$. Find the slope and inclination at $x=10$. (Why negative?)

10. An alternating current varied thus: $i=10\sin 200\,t$, where $200\,t$ is the phase angle in radians, and t is the number of seconds since starting. Find the rate at which i was increasing when $t=.002$.

11. Under the English income tax law recently, the balance (£y) of a man's income after paying his tax would vary with his original taxable income (£x) approximately thus: $y=x^{.974}+1.22\,x^{.962}-x$. Would y increase with x at $x=1\,000\,000$? Would it increase forever?

12. The energy transmitted by water flowing with speed v through a pipe line is expressible as $E=\pi a^2[cv-kv^3/ag]$. For what v is E a maximum?

13. To determine the best diameter of the nozzle in running an impulse wheel by a jet of water, it is necessary to find what value of x will minimize the expression $y=(1+kx^4)\div x^{\frac{4}{3}}$. Do this.

14. Differentiate in two different ways and check:

(*a*) $y=5\,x^2(x^3-20)$, (*b*) $y=\dfrac{x^3-20}{5\,x^2}$,

(*c*) $y=\log x^{20}$, (*d*) $y=\log\sqrt{x^5}$.

15. Differentiate:

(*a*) $e^{2x}(3\sin 3\,x+2\cos 3\,x)$, (*b*) $e^{-6t}(3\sin 2\,t+\cos 2\,t)$,

(*c*) $x^2\log x$, (*d*) $\cos\theta+\theta\sin\theta$,

(*e*) $\dfrac{kx}{x^2+a^2}$, (*f*) $\dfrac{\log x}{x^3}$,

(*g*) $\log\dfrac{x}{x^2+1}$, (*h*) $\log x^5+6\log\sqrt{x}+2\log\dfrac{3}{x^2}$,

(*i*) $\log(\log x)$, (*j*) $e^{2x}+2\,e^x+\log(e^x-1)^2+e^2$,

(*k*) $\log_{10}\sqrt{\dfrac{x^6-1}{x^6+1}}$, (*l*) $\log\sqrt[4]{x-2}-\dfrac{x-1}{(x-2)^2}$.

16. If $y=\frac{5}{3}\sqrt{x^2+7\,x+9}$, show that $\dfrac{dy}{dx}=\dfrac{5(2\,x+7)}{6\sqrt{x^2+7\,x+9}}$. Note the steps of the reduction carefully. Practice writing the derivatives of the following in final reduced form without intermediate steps:

$\sqrt{x^4-9\,x}$, $\frac{2}{3}\sqrt{25-x^2}$, $\frac{k}{c}\sqrt{ax+b}$, $\sqrt{2\,mx}$.

17. Differentiate:

(*a*) $(e^{3x}+1)^{10}$, (*b*) $(\sin 4t-\cos 4t)^6$,

(*c*) $\sqrt{e^{4x}-e^{-4x}}$, (*d*) $\sqrt{9+\sin^2\theta}$,

(*e*) $\log(x^2+\sqrt{x^4-1})$, (*f*) $\log_{10}(x+\sqrt{x^2+a^2})$,

(*g*) $\log x^2+\log\sqrt{x^4-1}$, (*h*) $a\log(\sqrt{x}+\sqrt{x+a})+\sqrt{ax+x^2}$.

18. In a certain type of chemical reaction the weight (x gm.) of substance formed varies thus with the elapsed time t sec.:

$$\log\frac{x}{a-x}=ka(t-c).$$

Show that the "speed of the reaction" (dx/dt) at any time equals $kx(a-x)$. For what value of x is this greatest?

19. Experiments indicate that the number of food calories consumed by an animal in growing to any weight (x kg.) is $F=x[a+b\log x]$. Show that the rate of consumption, per kg. gained, is a constant plus the instantaneous ratio of F to x.

20. The distance traveled by a point t min. after starting was $y=12t^3-t^4$. When was the acceleration greatest, and how great then?

21. The adjacent table shows the probable quantity (Q units) of a certain patented article that can be sold if the price per unit is x cents. The cost of production and marketing is $(4000+.03\,Q)$ dollars. What price will yield the maximum profit?

x	Q
2	500 000
4	400 000
6	300 000
8	200 000
10	100 000

22. If $y=x^3+5x+8$, find dy/dx and also dx/dy where $x=2$.

23. If $x=y+\log y$, find dy/dx where $y=1$.

24. If $x=10(\phi-\sin\phi)$, find $d\phi/dx$ at $\phi=\frac{\pi}{3}$.

§ 29. Indirect Dependence. If y depends upon u, and u upon x, then ultimately y depends upon x, though indirectly. To find dy/dx in such cases, we have by *Intro.*, § 76:

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}. \tag{13}$$

Or, if y is expressed in terms of x and we differentiate with respect to a third variable t, then

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}.$$

That is, we take the derivative (dy/dx) found directly from the equation involving y and x, and multiply it by dx/dt in order to get dy/dt.

Again, when a curve is given in parametric form:

$$x=f_1(t), \qquad y=f_2(t), \tag{14}$$

we can find dx/dt and dy/dt, and then get the slope dy/dx:

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}. \tag{15}$$

To find the flexion d^2y/dx^2 we differentiate this again, *with respect to* x.

Ex. I. Find the flexion at any point of the cycloid:

$$x=a(t-\sin t), \qquad y=a(1-\cos t).$$

Here $$\frac{dx}{dt}=a(1-\cos t), \qquad \frac{dy}{dt}=a\sin t. \tag{16}$$

By (15): $$\frac{dy}{dx}=\frac{\sin t}{1-\cos t}. \tag{17}$$

Differentiating this fraction *with respect to* x gives:

$$\frac{d^2y}{dx^2}=\frac{(1-\cos t)\left[\cos t\,\dfrac{dt}{dx}\right]-\sin t\left[\sin t\,\dfrac{dt}{dx}\right]}{(1-\cos t)^2}.$$

Note the common factor dt/dx. Using (16) we replace dt/dx by $1\div[a(1-\cos t)]$. Multiplying out the other factors:

$$\frac{d^2y}{dx^2}=\frac{\cos t-\cos^2 t-\sin^2 t}{a(1-\cos t)^3}=-\frac{1}{a(1-\cos t)^2}.$$

Substituting any value for t will give the desired flexion.

Ex. II. A spherical balloon expands. Find the relation between the rates at which the volume and radius are increasing per hour.

$$V=\tfrac{4}{3}\pi r^3.$$

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}. \tag{18}$$

This is the desired relation between dV/dt and dr/dt at any instant.

Another reason why dr/dt must appear in (18) is this: The rates at which V and r change per hour are related definitely; and the value of either at any instant determines the other. Hence an equation involving either dV/dt or dr/dt must involve the other also.

§ 30. Implicit Differentiation. Any equation involving x and y requires y to vary with x in some definite way. If the value of y is not given in terms of x explicitly, *i.e.*, in the form $y=f(x)$, we say that the equation defines y "implicitly" as a function of x; or that y is an *implicit function.*

To find dy/dx in such a case, we differentiate every term of the given equation *with respect to* x, collect terms, and solve for dy/dx algebraically.

E.g., if
$$x^2+y^2=10\,xy+a^2,$$
then
$$2\,x+2\,y\frac{dy}{dx}=10\left(x\frac{dy}{dx}+y\right). \tag{19}$$

[Note especially the derivative of y^2 with respect to x; also the derivative of the product xy.] Collecting terms:

$$(2\,y-10\,x)\frac{dy}{dx}=10\,y-2\,x.$$

$$\therefore \qquad \frac{dy}{dx}=\frac{5\,y-x}{y-5\,x}. \tag{20}$$

Remark. To find d^2y/dx^2 we would differentiate again *with respect to x throughout.* Notice the procedure carefully.

$$\frac{d^2y}{dx^2}=\frac{(y-5\,x)\left[5\frac{dy}{dx}-1\right]-(5\,y-x)\left[\frac{dy}{dx}-5\right]}{(y-5\,x)^2}.$$

Replacing each dy/dx by its value in (20) and simplifying:

$$\frac{d^2y}{dx^2}=\frac{(y-5\,x)\left[\frac{24\,y}{y-5\,x}\right]-(5\,y-x)\left[\frac{24\,x}{y-5\,x}\right]}{(y-5\,x)^2}.$$

Multiplying out, and combining further:

$$\frac{d^2y}{dx^2}=\frac{24(y^2+x^2-10\,xy)}{(y-5\,x)^3}. \tag{21}$$

But, by the original equation, the quantity $y^2+x^2-10\,xy=a^2$.

$$\therefore \qquad \frac{d^2y}{dx^2}=\frac{24\,a^2}{(y-5\,x)^3}. \tag{22}$$

§ 31. Logarithmic Differentiation. Usually the easiest way to differentiate a complicated product, fraction, power, root, or exponential function, is to *introduce logarithms* and simplify before differentiating.

Ex. I. $$y=\frac{(4\,x+3)^{\frac{2}{3}}}{(7\,x+1)^{\frac{3}{5}}}.$$

Here $$\log y=\tfrac{2}{3}\log(4\,x+3)-\tfrac{3}{5}\log(7\,x+1).$$

$$\therefore \frac{1}{y}\frac{dy}{dx}=\frac{8}{3(4\,x+3)}-\frac{21}{5(7\,x+1)}.$$

After combining the fractions in the right member, we multiply through by y, and then replace y by its value above:

$$\frac{dy}{dx}=\frac{28\,x-149}{15(4\,x+3)(7\,x+1)}\frac{(4\,x+3)^{\frac{2}{3}}}{(7\,x+1)^{\frac{3}{5}}}=\frac{28\,x-149}{15(4\,x+3)^{\frac{1}{3}}(7\,x+1)^{\frac{8}{5}}}.$$

Remark. Each binomial in the original quantity y occurs in dy/dx with its exponent lowered one unit: $-\frac{1}{3}$ in place of $+\frac{2}{3}$, and $-\frac{8}{5}$ in place of $-\frac{3}{5}$. Reflection upon this method of differentiating will show that a similar statement is true in general. (Cf. *Intro.*, § 179.)

EXERCISES

1. Find dy/dx if $y=\sqrt{u^2+5\,u+6}$ and $u=\log(x^2-3\,x+7)$; also if $y=t^3+t+5$ and $x=e^t-4\,t$.

2. Find the slope and flexion of the curve $x=t^3$, $y=t^2$, at the point where $t=2$.

3. The same as Ex. 2 for the curve $x=t^4+t$, $y=t^3+t$.

4. Find the slope of the circle $x = a\cos\phi$, $y = a\sin\phi$, at the point where $\phi = .3$. Likewise for the ellipse $x = a\cos\phi$, $y = b\sin\phi$.

5. Find the slope of the curve $x = a\cos^3\phi$, $y = a\sin^3\phi$, at the point where $\phi = \frac{3\pi}{4}$. What curve is this?

6. Find dy/dx in each of the following cases:

(a) $x = y^3 + y$, (b) $\sin x + \sin y = 1$, (c) $\log x + xy = e^y$.

7. (a), (b). Find d^2y/dx^2 in Ex. 6 (a), (b), respectively.

8. Find the slope and flexion of the curve $x^3 + y^3 = 9(x + y)$: (a) at the origin; (b) at one other point where $y = x$.

9. Find dy/dx in each following case:

(a) $y = (\sin x)^{\cos x}$, (b) $y = (5x)^{x^2}$, (c) $y = x^{e^x}$,

(d) $y = \sqrt{\dfrac{x^2 - 9}{x^2 + 9}}$, (e) $y = \dfrac{(x+9)^{\frac{3}{4}}}{(x+8)^{\frac{2}{3}}}$, (f) $y = \dfrac{(x+4)^7}{(x+8)^3(x+1)^4}$.

10. If y increases thus with x: $y = 2^{.1x}$, find its percentage rate of growth at any instant, *i.e.*, $(dy/dx) \div y$.

11. For any power law, $y = kx^n$, show that the percentage rate of growth varies inversely as x.

12. The "equilibrium constant" K of a balanced chemical reaction changes thus with the absolute temperature T:

$$K = K_0 e^{-\frac{q}{2}\frac{T - T_0}{T_0 T}},$$

where K_0, q, and T_0 have fixed values. Show that the percentage rate of change of K, per degree, varies inversely as T^2.

13. For a certain group of infants the relation between the average weight (x oz.) and the age (t mo.) was found to be approximately:

$$\log_{10}\frac{x}{341.5 - x} = .136(t - 1.66).$$

Show that the rate of growth at any time (for an average infant) was $.00092\,x(341.5 - x)$. [This will be discussed further in § 195.]

14. Proceed as in Ex. 13 for another group for whom

$$\log_{10}\frac{x}{350 - x} = .111(t - 2.47).$$

15. Differentiate: $e^{\sqrt{\frac{1+\sin x}{1-\sin x}}}\left(\sqrt{\dfrac{1+\sin x}{1-\sin x}} - 1\right)$. [Hint: Let the radical equal u temporarily; and apply § 31 to that equality, to get du/dx.]

16. Differentiate by any method:

(*a*) $e^{3x}(3x^3-3x^2+2x-\frac{2}{3})$, (*b*) $2x\cos x+(x^2-2)\sin x$,

(*c*) $\log_{10}\sqrt[3]{\frac{1-x^3}{1+x^3}}$, (*d*) $\frac{x\log x}{1-x}+\log(1-x)$,

(*e*) $\log\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$, (*f*) $\sqrt{1-x^2}+\log\frac{x}{1+\sqrt{1-x^2}}$.

17. Has the slope of a line any linear dimensions, or is it a pure number? How about $\Delta y/\Delta x$ and dy/dx? Is the formula $l=dy/dx$ homogeneous? What linear dimensions has dl/dx or d^2y/dx^2? Apply the homogeneity test to equations (18)–(22).

PART II. FURTHER DIFFERENTIATION

§ 32. Basic Trigonometric Functions. Differentiation formulas for the tangent, cotangent, secant, and cosecant can be found by expressing these functions in terms of the sine and cosine, and then differentiating by the fraction rule. (Cf. *Intro.*, § 273.) Thus we obtain

$$\frac{d}{dx}(\tan u)=\quad\sec^2 u\,\frac{du}{dx},\tag{23}$$

$$\frac{d}{dx}(\operatorname{ctn} u)=-\csc^2 u\,\frac{du}{dx},\tag{24}$$

$$\frac{d}{dx}(\sec u)=\quad\sec u\tan u\,\frac{du}{dx},\tag{25}$$

$$\frac{d}{dx}(\csc u)=-\csc u\operatorname{ctn} u\,\frac{du}{dx}.\tag{26}$$

These formulas are correct for angles in radians. For degree measure the results should be multiplied by $\pi/180$, or .017453, approx.

Memorize carefully the verbal statements of these formulas; *e.g.*, *the derivative of the tangent of any angle is the secant squared, times the derivative of the angle.*

Give special attention to formulas (23) and (25); but notice how similarly the co-functions run, though with negative signs.

§ 33. Two Auxiliary Trigonometric Functions. In engineering, particularly in laying out railway curves, two auxiliary functions, called the *versine* and the *exsecant*, are considerably used. They are defined thus:

$$\text{vers}\,\theta = 1 - \cos\theta, \qquad \text{exsec}\,\theta = \sec\theta - 1. \tag{27}$$

The utility of these functions can be seen from Fig. 21. Here $c = r\cos\theta$, and $OS = r\sec\theta$.

Hence
$$v = r - c \quad = r(1 - \cos\theta) = r\,\text{vers}\,\theta,$$
$$e = OS - r = r(\sec\theta - 1) = r\,\text{exsec}\,\theta.$$

Thus a table of versines can be used to determine the correct distance of a curved track from its chord at the middle; and a table of exsecants to determine the deflection of the track at its middle from a tangent at one end.

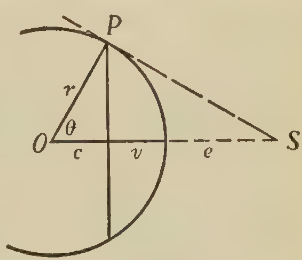


FIG. 21.

By remembering the definitions of these functions we can write their derivatives at once, without any special rule:

$$\frac{d}{dx}(\text{vers}\,u) = \sin u\frac{du}{dx};$$

$$\frac{d}{dx}(\text{exsec}\,u) = \sec u \tan u\frac{du}{dx}\cdot]$$

For degree measure multiply by .017453.

EXERCISES

1. Differentiate with respect to the variable involved:

(*a*) $\frac{1}{2}\tan(6t + .4) - 3t$, (*b*) $5\theta + \frac{1}{4}\text{ctn}(20\theta - .7)$,

(*c*) $5\tan 2x + 4\sec 2x$, (*d*) $7\,\text{ctn}\,3\phi - \csc 3\phi$,

(*e*) $\sin^3 2\theta + \cos^3 2\theta$, (*f*) $\sec^3\frac{\theta}{3} + 3\tan\frac{\theta}{3}$.

2. Find the second derivative of each of the following:

(*a*) $2 \sin^6 4\theta$, (*b*) $\frac{1}{8} \sec^4 2t$, (*c*) $\operatorname{ctn}^2\left(\frac{x}{2}+\frac{\pi}{4}\right)$,

(*d*) $\log \csc \phi$, (*e*) $\log(\sec\theta + \tan\theta)$, (*f*) $\log \sin^2 x$.

3. Find the third derivative of each of the following:

(*a*) $k \tan 2x$, (*b*) $12 \sec\theta$, (*c*) $4 \csc\phi$.

4. Differentiate:

(*a*) $\operatorname{vers}\theta + \operatorname{exsec}\theta$, (*b*) $6 \operatorname{vers}\left(\frac{t}{3}+.2\right)$,

(*c*) $20 \operatorname{exsec}\left(\frac{x}{4}+\frac{\pi}{4}\right)$, (*d*) $\frac{1}{3}\operatorname{ctn}^3\theta - \operatorname{ctn}\theta - \theta$,

(*e*) $\tan^3\frac{\theta}{3} - 3\tan\frac{\theta}{3} + \theta$, (*f*) $e^{\tan\theta}\sec\theta$,

(*g*) $\dfrac{a(1-e^2)}{1-e\cos\theta}$, (*h*) $\dfrac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$,

(*i*) $4\cos\theta + \sqrt{16 - 9\sin^2\theta}$.

5. Simplify and then differentiate:

(*a*) $\dfrac{\sin^2\phi - \cos^2\phi}{\tan\phi - \operatorname{ctn}\phi}$, (*b*) $\dfrac{\tan^2 x}{\sec x + 1}$,

(*c*) $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta}$, (*d*) $\dfrac{\tan\theta + \operatorname{ctn}\theta}{\sec\theta\csc\theta}$,

(*e*) $\dfrac{\sin 2x}{\cos x}$, (*f*) $\operatorname{ctn}\theta - \dfrac{\cos 2\theta}{\sin\theta\cos\theta}$.

6. How fast does the slope of a curve change, per degree change in the inclination, at a point where $I = 30°$? Where $I = 60°$?

7. A rope is pulled through a hole in a ceiling 20 ft. high, and the end is carried horizontally in a line 5 ft. high, keeping the rope taut as pulled out. How fast is the length outside increasing, per unit decrease in the inclination, when $I = 60°$?

8. How fast does r change with θ, per degree, in the three-leaved rose $r = 15 \sin 3\theta$, at $\theta = 15°$?

9. The same as Ex. 8 for the lemniscate $r^2 = 100 \cos 2\theta$.

10. The distance of a comet from the sun varies thus: $r = k \sec^2\left(\frac{\theta}{2}\right)$. Find the rate of change of r, per radian, at $\theta = \frac{\pi}{3}$.

11. The upsetting moment for a retaining wall of a certain type depends thus upon ϕ, the angle of repose of the earth: $M =$

$\frac{1}{6}\,wh^3\tan^2\left(45°-\frac{\phi}{2}\right)$, where w and h are constants. Find the rate of change of M with ϕ per degree.

12. If a ship sails constantly 30° north of east, its longitude θ will vary thus with its latitude L: $\theta=\sqrt{3}\log\text{ctn}\left(\frac{\pi}{4}-\frac{L}{2}\right)+C$. How fast does θ increase with L per radian at $L=\frac{\pi}{3}$?

§ 34. Exponential Functions. Any constant raised to any variable power is called an exponential function. It may be written

$$y=a^u. \tag{28}$$

Differentiating logarithmically with respect to x, we find

$$\frac{dy}{dx}=a^u\log a\frac{du}{dx}. \tag{29}$$

That is, *the derivative of a constant with a variable exponent is the same quantity times the logarithm of the constant* (to the base e) *times the derivative of the exponent.*

If the constant a happens to be e, then $\log a=1$, and (29) reduces to the familiar derivative of e^u, viz. $e^u du/dx$. The only new feature in (29) is the factor $\log a$, which was not written in the e^u formula because merely 1 in that case.

Ex. I. Differentiate $10^{\sin x}$.
Answer: $10^{\sin x}\log 10\cos x$, $=2.3026\cos x\,10^{\sin x}$.

EXERCISES

1. Differentiate: 2^{x^2}, 5^{-6x}, $10^{.7x}$, $a^{-.02x+3}$, k^{e^x}, 5^{2^x}.

2. Differentiate, either directly or logarithmically, as preferred:

(*a*) $\log(2^x+2^{-x})$, (*b*) $\log_{10}(2^x+2^{-x})$, (*c*) $\log(10^x\cdot x^{10})$,

(*d*) $\dfrac{3^x-3^{-x}}{3^x+3^{-x}}$, (*e*) $\sqrt{5^{\sin\theta}+1}$, (*f*) $\log x^{\frac{1}{x}}$,

(*g*) $4^{\text{ctn}\,\theta}$, (*h*) $x^x\cdot 10^{\csc x}$, (*i*) $\log\sec 10^{x^2}$.

3. Differentiate directly; also after multiplying or dividing out:

(*a*) $(e^{5x}+e^{-5x})^2$, (*b*) $\dfrac{e^{4x}+1}{e^{2x}}$, (*c*) $e^{2x}(e^{3x}-5\,e^{-x}-e^{-3x})$.

4. Find the second derivative of each of the following:

(*a*) $10^{3x} \cdot e^{2x}$, (*b*) $\log(4+6^x)$, (*c*) $\log(e^{2x}+e^{-2x})$,
(*d*) $10^{\tan\theta}$, (*e*) $10^t \sin t$, (*f*) $4^x \cdot x^4$.

5. Show that $y=k \cdot b^x$ is a case of the Compound Interest Law: (*a*) by finding the percentage rate of change; (*b*) by considering b as some power of e.

6. The weight (y gm.) of an invertebrate animal t days after beginning a fast was: $y=k(.9433)^t$. Find the percentage rate of decrease at any time.

7. For a large group of British boys and men the average weight (x kg.) at any age (t yr.) between $3\frac{1}{2}$ and $30\frac{1}{2}$ was approximately:

$$x=9+24\frac{e^{.188(t-5.5)}}{1+e^{.188(t-5.5)}}+35\frac{e^{.272(t-16)}}{1+e^{.272(t-16)}}.$$

Find the rate of growth in weight at $t=10.5$. (Cf. § 195 later.)

8. The same as Ex. 7 for a Belgian group, where

$$x=6.2+18.7\frac{e^{.169(t-5.5)}}{1+e^{.169(t-5.5)}}+41.4\frac{e^{.175(t-16.5)}}{1+e^{.175(t-16.5)}}.$$

9. Gompertz's mortality law gives the number of survivors (l_x) at any age (x yr.) out of an original group as $l_x=kg^{c^x}$, where k, g, and c are constants. Find the percentage death rate at any age; *i.e.*,

$$-(dl_x/dx)\div l_x.$$

10. The same as Ex. 9 for Makeham's law: $l_x=ks^xg^{c^x}$, where s also is a constant.

§ 35. Inverse Functions. If y is any function of x, then x is called the *inverse function* of y. For instance,

$$\text{if} \quad y=x^3, \qquad \text{then} \quad x=\sqrt[3]{y},$$

so that the cube and the cube root are mutually inverse functions. Likewise the exponential function and the logarithm are mutually inverse. For

$$\text{if} \quad y=e^x, \qquad \text{then} \quad x=\log y.$$

The inverses of the trigonometric functions are important in many calculations, — especially the inverses of the sine and tangent, now to be considered.

§ 36. The Arcsine and Arctangent. The symbol $sin^{-1}x$ is not used to denote a negative power of sin x as its form

would suggest, but is used instead to denote *an angle whose sine is x.** There are infinitely many such angles for any value of x between -1 and $+1$; and to be definite we shall mean always *the angle nearest zero*, — and expressed in radians. Thus if we write

$$y = \sin^{-1} x, \qquad (30)$$

we shall mean that y is the number of radians in that particular angle between $-\frac{\pi}{2}^{(r)}$ and $\frac{\pi}{2}^{(r)}$ whose sine is x. This is the inverse of $x = \sin y$.

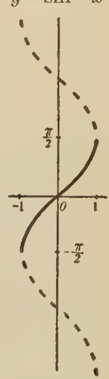


FIG. 22.

If x varies, y or $\sin^{-1} x$ varies with it, as shown in Fig. 22. (This is evidently the sine curve reversed as to x and y, — *i.e.*, reflected in the line $y = x$.) Using only the value of y nearest zero, we confine ourselves to the "principal" part of the graph, nearest the origin, which is drawn full in Fig. 22. This part rises throughout, from $x = -1$ to $x = 1$. Hence the derivative of $\sin^{-1} x$ is positive.

Another notation for $\sin^{-1} x$ is $\arcsin x$. Either is read "the arcsine of x," or "the inverse sine of x," as well as "an angle whose sine is x."

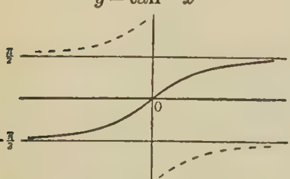


FIG. 23.

Similar statements apply to the function $\tan^{-1} x$ or $\arctan x$, and to its graph. (Fig. 23.)

Derivatives. We shall often need to differentiate an arcsine or arctangent, say $\sin^{-1} u$ or $\tan^{-1} u$, where u denotes some function of x. The formulas are derived as follows:

If $\qquad y = \sin^{-1} u,$
then $\qquad u = \sin y.$

* The minus first power of $\sin x$ is written $(\sin x)^{-1}$, an exception to the usual trigonometric power notation. Cf. *Intro.*, §§ 266–7.

We desire dy/dx. Differentiating with respect to x:

$$\frac{du}{dx} = \cos y \frac{dy}{dx}.$$

$$\therefore \qquad \frac{dy}{dx} = \frac{\frac{du}{dx}}{\cos y}. \tag{31}$$

But as $\sin y = u$, hence $\cos y = \sqrt{1-u^2}$, giving finally

$$\boldsymbol{\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} u) = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}}. \tag{32}$$

The positive sign of the radical is taken since $\sin^{-1} u$ increases with u, as noted above.

Thus, *the derivative of the arcsine of any quantity equals the derivative of that quantity, divided by the square root of one minus the square of the quantity.*

Likewise, if $y = \tan^{-1} u$, we find that

$$\frac{dy}{dx} = \frac{\frac{du}{dx}}{\sec^2 y}.$$

But as $\tan y = u$, hence $\sec^2 y = 1 + u^2$, and we get

$$\boldsymbol{\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} u) = \frac{\frac{du}{dx}}{1+u^2}}. \tag{33}$$

Memorize carefully the verbal statement of (32) above and a similar statement for (33).

Ex. I. $$\frac{d}{dx} \sin^{-1} x^3 = \frac{3x^2}{\sqrt{1-x^6}}.$$

Ex. II. $$\frac{d}{dx} \arctan \frac{x}{6} = \frac{\frac{1}{6}}{1 + \frac{x^2}{36}} = \frac{6}{36 + x^2}.$$

EXERCISES

1. Look up or find otherwise the (principal) values of:

(*a*) $\sin^{-1} .5$, (*b*) $\sin^{-1}(-1)$, (*c*) $\sin^{-1} 0$, (*d*) $\sin^{-1}(-.8562)$,
(*e*) $\tan^{-1} .5$, (*f*) $\tan^{-1}(-1)$, (*g*) $\tan^{-1} 0$, (*h*) $\tan^{-1}\sqrt{3}$.

2. When a simple pendulum makes very short swings, the time elapsed from the lowest point to any displacement angle θ is approximately: $t = \sqrt{l/g}\,\arcsin(\theta/a)$. Express θ in terms of t.

3. If $\tan\dfrac{x}{2} = u$, express x and dx/du in terms of u.

4. Differentiate and simplify the results:

(*a*) $\arcsin 7x$, (*b*) $\arctan x^4$, (*c*) $\arcsin(1-2x^2)$,

(*d*) $\sin^{-1}\dfrac{x}{5}$, (*e*) $\sin^{-1}\dfrac{x}{a}+C$, (*f*) $\tan^{-1}\dfrac{x}{a}+k$,

(*g*) $\tan^{-1}\dfrac{2x}{1-x^2}$, (*h*) $\sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$, (*i*) $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$.

5. Differentiate and simplify results:

(*a*) $x\sqrt{4-x^2}+4\sin^{-1}\dfrac{x}{2}$, (*b*) $x\arcsin x-\sqrt{1-x^2}$,

(*c*) $\sin^{-1}\dfrac{1}{x}+\tan^{-1}\sqrt{x^2-1}$, (*d*) $x\arctan\dfrac{x}{3}-\log(9+x^2)^{\frac{3}{2}}$,

(*e*) $\log\dfrac{\sqrt{1+x^2}}{1+x}+\tan^{-1}x$, (*f*) $(x^2+1)10^{\tan^{-1}x}$,

(*g*) $\sin^{-1}(\frac{3}{5}\cos\theta)$, (*h*) $\tan^{-1}(\sqrt{\frac{3}{2}}\tan\theta)$,

(*i*) $\tan^2\theta-\log\sec^2\theta$, (*j*) $(x-1)^{\frac{5}{3}}(x+4)^{\frac{2}{3}}(x+5)^2$.

6. The polarizing angle for a substance with index of refraction n is $A = \arctan n$. At what rate does A change, per unit change in n, when $n = \frac{4}{3}$?

7. The angle of diffraction in the second spectrum produced by light of a wave length λ microns, striking a grating with rulings 1.2 microns apart at an incidence angle of 30°, is $D = \arcsin(\frac{5}{3}\lambda - .5)$. At what rate does D change with λ, when $\lambda = .6$?

8. How fast does the inclination of a curve change, per unit change in the slope l, when $l = 2$?

9. A balloon rises vertically, starting from a point A on level ground. If observed from a point B on the ground 1000 ft. from A, how fast will the angle of elevation be increasing, per foot rise, when the height is 600 ft.?

10. At what point on the positive X-axis would the portion of the Y-axis between (0, 2) and (0, 7) subtend the largest angle?

§ 37. Other Inverse Trigonometric Functions. $\mathrm{Cos}^{-1} x$, $\mathrm{ctn}^{-1} x$, $\sec^{-1} x$, $\csc^{-1} x$, and $\mathrm{vers}^{-1} x$ are defined in a manner analogous to that used for $\sin^{-1} x$ and $\tan^{-1} x$ in § 36. The principal value of each is agreed upon as shown by the full line in each graph of Fig. 24 below, making each function

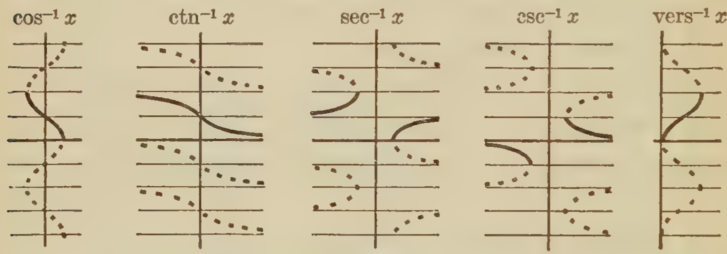


FIG. 24.

either increase throughout or decrease throughout the interval for which the function has a real value.

These graphs, which are the graphs of the corresponding trigonometric functions reflected in the line $y=x$, need not be memorized; but reference to them here will occasionally be helpful, in choosing a "principal value" correctly.

Derivatives. Differentiation formulas for all these inverse functions are found in the same way as for $\sin^{-1} u$ and $\tan^{-1} u$ in § 36. The hardest case will be worked out here, and the rest left as exercises.

Let $\qquad y=\mathrm{vers}^{-1} u.$

Then $\qquad u=\mathrm{vers}\, y=1-\cos y. \qquad (34)$

$$\frac{du}{dx}=\sin y\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{\frac{du}{dx}}{\sin y}$$

But by (34): $\cos y = 1 - u$; whence $\sin y = \sqrt{1-(1-u)^2} = \sqrt{2u-u^2}$.

$$\therefore \qquad \frac{dy}{dx} = \frac{\frac{du}{dx}}{\sqrt{2u-u^2}}. \tag{35}$$

Ex. I. $\frac{d}{dx}(\text{vers}^{-1}\, 9x^2) = \frac{18x}{\sqrt{18x^2-81x^4}} = \frac{6}{\sqrt{2-9x^2}}.$

Ex. II. $\frac{d}{dx}\left(\text{vers}^{-1}\frac{x}{5}\right) = \frac{\frac{1}{5}}{\sqrt{\frac{2x}{5}-\frac{x^2}{25}}} = \frac{1}{\sqrt{10x-x^2}}.$

The complete list follows; this should be memorized. Concentrate upon the formulas in boldface type, but notice how the others differ.*

$$\boldsymbol{\frac{d}{dx}(\sin^{-1}u) = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}}, \tag{36}$$

$$\frac{d}{dx}(\cos^{-1}u) = -\frac{\frac{du}{dx}}{\sqrt{1-u^2}}, \tag{37}$$

$$\boldsymbol{\frac{d}{dx}(\tan^{-1}u) = \frac{\frac{du}{dx}}{1+u^2}}, \tag{38}$$

$$\frac{d}{dx}(\text{ctn}^{-1}u) = -\frac{\frac{du}{dx}}{1+u^2}, \tag{39}$$

$$\boldsymbol{\frac{d}{dx}(\sec^{-1}u) = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}}}, \tag{40}$$

$$\frac{d}{dx}(\csc^{-1}u) = -\frac{\frac{du}{dx}}{u\sqrt{u^2-1}}, \tag{41}$$

$$\boldsymbol{\frac{d}{dx}(\text{vers}^{-1}u) = \frac{\frac{du}{dx}}{\sqrt{2u-u^2}}}. \tag{42}$$

* See the remark at the top of the next page.

In formulas (40) and (41) the sign must be changed when u is negative. For $\sec^{-1} u$ is an increasing function of u, along the "principal part" of the graph, even when u is negative. (Fig. 24.) And similarly $\csc^{-1} u$ is a decreasing function.

§ 38. Classification of Functions. We have now differentiated all the basic functions: power, exponential, logarithmic, trigonometric, and inverse trigonometric. These are often classified into two groups: Algebraic Functions, and Transcendental Functions.

A function y is called algebraic if it could be a root of some equation of the form

$$y^n + f_1(x)y^{n-1} + f_2(x)y^{n-2} \cdots + f_n(x) = 0, \qquad (43)$$

where n is a positive integer and the f's are rational integral ("polynomial") functions of x. Algebraic functions include powers with constant exponents and coefficients; also combinations of such powers formed by a finite number of additions, multiplications, divisions, and extractions of roots; also implicit functions defined by equations such as (43) but not expressible explicitly by means of radicals or powers.

EXERCISES

1. Find the (principal) values of the following:

(a) $\cos^{-1} 0$,	$\cos^{-1}(-.5)$,	$\cos^{-1} 1$;
(b) $\operatorname{ctn}^{-1} .77$,	$\operatorname{ctn}^{-1} 1$,	$\operatorname{ctn}^{-1}(-1)$;
(c) $\sec^{-1} 2$,	$\sec^{-1}\sqrt{2}$,	$\sec^{-1}(-1)$;
(d) $\csc^{-1} 2$,	$\csc^{-1}(-\sqrt{2})$,	$\csc^{-1}(-1)$.

2. In a certain type of oscillation the time elapsed and the distance x from a central point are related thus: $t + c = \cos^{-1}(x/a)$. Express x as a function of t. Expand in terms of $\sin t$ and $\cos t$.

3. Calculations on planetary motion give the polar angle θ in terms of the radius vector r in the form

$$\theta = c + \cos^{-1}\left(\frac{k}{r} - a\right).$$

Express r as a function of θ.

4. From the parametric equations of the cycloid: $x=a(\phi-\sin\phi)$, $y=a(1-\cos\phi)$, express ϕ as a function of y and then eliminate ϕ to obtain the rectangular equation:

$$x=a\operatorname{vers}^{-1}\frac{y}{a}-\sqrt{2ay-y^2}.$$

5. Differentiate:

(a) $\cos^{-1}x^3$, (b) $\operatorname{ctn}^{-1}6x$, (c) $\sec^{-1}2x$,

(d) $\operatorname{vers}^{-1}x^4$, (e) $\cos^{-1}\left(\dfrac{1}{2x^7}\right)$, (f) $\sec^{-1}2x^7$,

(g) $\csc^{-1}\left(\dfrac{4}{x}\right)$, (h) $\sin^{-1}\left(\dfrac{x}{4}\right)$, (i) $\cos^{-1}\sqrt{1-x^2}$,

(j) $\sec^{-1}\sqrt{1+x^2}$, (k) $\operatorname{ctn}^{-1}\dfrac{1}{\sqrt{x^2-1}}$, (l) $\operatorname{vers}^{-1}2x^2$.

6. Differentiate:

(a) $\arctan\dfrac{2}{x}+\log\sqrt{\dfrac{x+2}{x-2}}$, (b) $\operatorname{arcctn}\sqrt{\dfrac{1+\cos x}{1-\cos x}}$,

(c) $\dfrac{\sqrt{x^2-a^2}}{a}-\sec^{-1}\dfrac{x}{a}$, (d) $\cos^{-1}\dfrac{x}{a}-\dfrac{\sqrt{a^2-x^2}}{x}$,

(e) $\operatorname{vers}^{-1}\dfrac{x}{a}+\dfrac{\sqrt{2ax-x^2}}{a}$, (f) $\sec^{-1}x+\dfrac{\sqrt{x^2-1}}{x^2}$.

7. In Ex. 4 find dy/dx in terms of y; also find it in terms of ϕ and compare.

8. Derive formulas (37), (39), (40).

9. Show that $\cos^{-1}x+\cos^{-1}y=\cos^{-1}(xy-\sqrt{(1-x^2)(1-y^2)})$. [Hint: Let $\cos^{-1}x=\theta$ and $\cos^{-1}y=\phi$. The question, then, is whether the sum of θ and ϕ will be an angle whose cosine is the last given value.]

10. Show that $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$.

§ 39. Concerning Maxima and Minima. Two important facts relating to extreme values should be noted carefully at this point.

(I) *The greatest value reached may be merely an end-value.* A function y may increase to some greatest value and then suddenly cease to exist, or cease to fit the problem in which it was used. The maximum or greatest value in such a case is not a maximum in our technical sense of a turning value up to which y increases, and then decreases; and it cannot be found by using the derivative.

For instance, the greatest speed of a raindrop is attained at the instant it strikes the ground. The mathematical function which expresses the speed while falling does not properly reach any maximum, and would go on increasing with t. But, as far as a free fall is concerned, time does not continue beyond the instant of impact!

Unless otherwise stated we shall continue to use the word "maximum" in the turning sense. Hence, in seeking any absolute maximum or greatest value reached, we should not only find any turning maximum but also consider the graph of the function involved or think of boundary limitations, and see whether any greater value is reached at a terminus or end-point.

(II) *A maximum may be reached in several different ways*, as illustrated in Fig. 25. At A the derivative dy/dx becomes

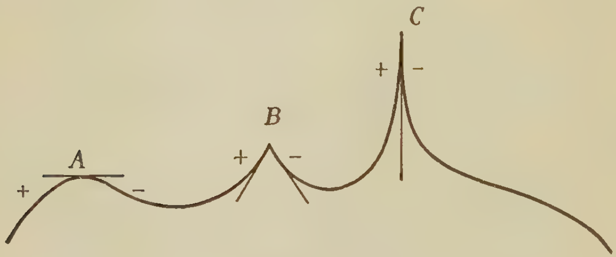


FIG. 25.

zero. At B, dy/dx suddenly jumps from some positive to some negative value, without becoming zero. At C, dy/dx makes an infinite jump, the slope having enormous $+$ values to the left of the vertical tangent and enormous $-$ values to the right. Note that y itself does not jump at C, but has a definite maximum value even though dy/dx jumps.

The one thing true at all the points A, B, C, is that dy/dx changes from $+$ to $-$. This is necessary for a maximum. It is also sufficient to insure a maximum if y has a finite value at the point in question.

Case B is very rare and is not likely to be encountered in any ordinary problem.

Case C may arise if y involves a factor or term with a positive fractional exponent. But no special rule is needed for that case: simply see whether dy/dx has a denominator which can be zero (making dy/dx infinite), while y remains finite.

Statements analogous to those above concerning maxima apply also to minima.

§ 40. **Implicit Method of Applying Conditions.** Heretofore, in making a given quantity a maximum or minimum, subject to some specified condition, we have utilized the condition to express the given quantity as a function of a *single variable* before differentiating.

This preliminary step, however, is not essential. We can adopt another procedure: one which is especially convenient in problems about *shapes* of figures.

To illustrate, let us find the most economical shape for a cylindrical can, — in other words, the shape which will make the total surface as small as possible for a given volume:

$$S = 2\pi r^2 + 2\pi r h. \qquad \text{(To be minimized)}$$

$$V = \pi r^2 h. \qquad \text{(To be constant)}$$

Differentiate both of these equations at once with respect to r. (And note that the products rh and r^2h give two terms each.)

$$\frac{dS}{dr} = 4\pi r + 2\pi h + 2\pi r\frac{dh}{dr} = 0, \tag{44}$$

$$\frac{dV}{dr} = 2\pi r h + \pi r^2\frac{dh}{dr} = 0. \tag{45}$$

(Why is $dS/dr = 0$? $dV/dr = 0$?) From (45) we find

$$\frac{dh}{dr} = -\frac{2h}{r}.$$

This substituted in (44) gives:

$$4\pi r + 2\pi h - 4\pi h = 0;$$

whence $h = 2r$. The height must be equal to the diameter.

Remark. If a quantity Q depends on several variables, with so few conditions imposed as to leave two or more variables independent, we use a later method. (§ 54.)

EXERCISES

1. A rectangle is to have the least perimeter for a given area. Show that it must be a square.

2. A cylindrical tank with no top is to have the least possible total area for a given volume. What ratio must the height bear to the radius?

3. The same as Ex. 2 for a conical tank.

4. Like Ex. 2 for a rectangular tank with a square base, of side x.

5. If a rectangular space of a certain area is to be inclosed by side fences costing \$.50 per yd. and end walls costing \$1.50 per yd., what relative dimensions would minimize the cost?

6. An uncovered cylindrical tank is to have a certain volume. The base will cost \$4 per sq. yd. and the wall \$3 per sq. yd. What ratio of height to radius would be most economical?

7. The same as Ex. 6 for a rectangular tank with a square base.

8. In a Norman window the sides and bottom are straight but the top is a semi-circle. To get the largest total area for a given perimeter, what should be the proportions of the rectangular part?

9. For the yellow band in a rainbow the deviation ($D°$) of light emerging from a raindrop after three internal reflections is: $D = 2(i - r) + 3(\pi - 2r)$, where i and r are the angles of incidence and reflection. Also $\sin i = \frac{4}{3}\sin r$. Find what i makes D a minimum.

10. Each month a factory makes and sells x units of a commodity, at an expense of \$$E$ which varies thus with x:

$$E = 40000 + 10x + .001x^2.$$

The price obtainable (\$$p$ per unit) also varies with x, viz. $p = 50 - .001x$. What x will make the entire profit (\$$P$) a maximum, and how great?

11. The height of a falling object t sec. after starting was $y = 1600 - 16t^2$. Find the greatest speed attained. Explain.

12. Find the least distance from $(-3, 0)$ to the curve $y^2 = 10x$, — first by inspection; then by taking D as $f(y)$. Try $f(x)$; cf. (I), p. 57.

In Ex. 13–18 find each maximum or minimum without differentiation, by using some fact about the sine function.

13. What is the maximum value of $10+4\sin\theta$, and at what angle is it reached? Also the minimum value?

14. The same as Ex. 13 for $7+2\sin 3\theta$.

15. Rewrite the quantity $20\sin\theta\cos\theta$ so as to see its maximum value by inspection. When a helical spring of pitch-angle θ is stretched, its end tends to rotate, proportionally to $\sin\theta\cos\theta$. What angle would maximize the rotation?

16. The time of flight of a projectile fired at an inclination θ is proportional to $\sin\theta$; the constant horizontal speed is proportional to $\cos\theta$. Hence what θ gives the maximum range? (This ignores air resistance.)

17. The magnetic declination ($D°$) at a certain city t yr. after Jan. 1, 1900, has varied thus: $D=5.27+3.05\sin(1.46\,t+38.2)$, all these angles being in degrees. Find the maximum D and when it will occur. Also find the rate at which D varied, at $t=20$.

18. A cable 20 ft. long is to be attached to a telephone pole and to a stake in the ground. For what inclination will the direct distance from the foot of the pole to the cable be greatest?

19. Check equations (44) and (45) by the homogeneity test.

§ 41. Studying a Curve by its Slope. In drawing a curve it is helpful to know the slope at various points, — especially at each point where the curve meets the X- or Y-axis. We can then draw the curve in the right direction through any such point. Some illustrations follow.

(*A*) *Cubical Parabola:* $y=kx^3$.

This lies below the X-axis to the left of $x=0$, and above to the right. It is tangent to the axis at the point of crossing, for the slope dy/dx is zero at $x=0$. (Fig. 26, *A*.)

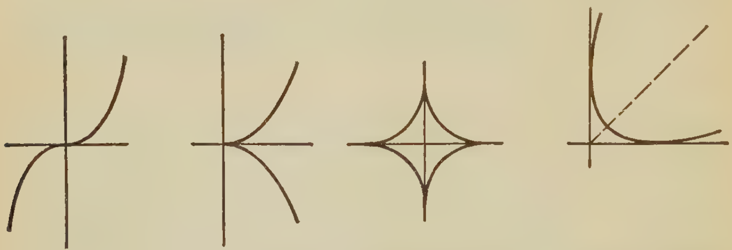
(*B*) *Semi-Cubical Parabola:* $y^2=kx^3$, or $y=\pm\sqrt{k}x^{\frac{3}{2}}$.

This also meets the axes at (0, 0). But it does not extend to the left, where x would be negative. To the right it has two symmetrical branches. (Cf. § 5.) Now

$$\frac{dy}{dx}=\pm\frac{3}{2}\sqrt{kx}.$$

Hence the slope at $x=0$ is zero. Thus the curve does not leave the origin vertically as the ordinary parabola does, but horizontally, forming a sharp "cusp." (Fig. 26, *B*.)

CUBICAL PARABOLA | SEMI-CUBICAL PARABOLA | HYPOCYCLOID OF 4 CUSPS | PARABOLA (TURNED 45°)



(*A*) $y=kx^3$ (*B*) $y^2=kx^3$ (*C*) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=10^{\frac{2}{3}}$ (*D*) $x^{\frac{1}{2}}+y^{\frac{1}{2}}=20^{\frac{1}{2}}$

FIG. 26.

(C) *Hypocycloid:* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=10^{\frac{2}{3}}$.

The curve meets the axes at $x=0$, $y=\pm 10$; and at $y=0$, $x=\pm 10$. It does not extend beyond these points; for $x^{\frac{2}{3}}$ and $y^{\frac{2}{3}}$ are both essentially positive and hence neither may exceed $10^{\frac{2}{3}}$.

Differentiating:

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0.$$

$$\therefore \qquad \frac{dy}{dx}=-\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}=-\frac{\sqrt[3]{y}}{\sqrt[3]{x}}. \qquad (46)$$

At $(\pm 10, 0)$, $dy/dx=0$ and the direction is horizontal. At $(0, \pm 10)$, $dy/dx=\infty$ and the direction is vertical. The curve must have four sharp cusps. (Fig. 26, *C*. See also §§ 21–22.)

(*D*) *Parabola:* $x^{\frac{1}{2}}+y^{\frac{1}{2}}=20^{\frac{1}{2}}$.

The curve is confined to the first quadrant where x and y are both positive; and, with the present signs of the radicals,

may not run beyond its intersections with the axes: (20, 0), (0, 20).

Differentiating gives $dy/dx = -\sqrt{y} \div \sqrt{x}$. Hence the direction at (20, 0) is horizontal, and at (0, 20) vertical (Fig. 26, *D*). The curve is in reality a portion of a parabola, tangent to the axes and turned 45°. (See Exs. 8–9, p. 377.) The rest of the parabola would be obtained by modifying the equation, to let $x^{\frac{1}{2}}$ or $y^{\frac{1}{2}}$ have a negative sign.

§ 42. Differentials. As shown in *Intro.*, § 83, a derivative dy/dx or $f'(x)$, though otherwise defined originally, can be regarded as a fraction $dy \div dx$. The two "differentials" dy and dx are simply any two quantities whose ratio equals $f'(x)$. Or,

$$\boldsymbol{dy = f'(x)\, dx.} \tag{47}$$

If we plot y as a function of x, dy will be represented by the distance the tangent line rises in a horizontal distance dx. For a small interval, dy is nearly the same as Δy, the rise of the graph itself.

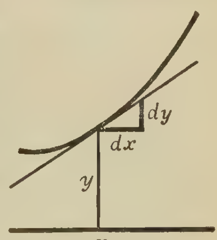


FIG. 27.

We may also say that dy is the approximate change in y produced by any very small change dx in the value of x. And by (47) we can find such an approximate change by simply multiplying the derivative $f'(x)$ by dx.

E.g., if $y = x^4$, and x changes from 2 to 2.00001, then the approximate change in y is

$$dy = 4\, x^3 dx = 4(8)(.00001) = .00032.$$

In other words, while a derivative gives a *rate of change*, a differential gives an approximate increment, or approximate *amount of change*.

Treating derivatives as fractions helps us to combine them easily. For instance, merely canceling dx will give

$$\frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dt}, \tag{48}$$

which is also known to be correct by the rule covering Indirect Dependence. (§ 29.)

Similar operations, however, are not possible for higher derivatives:

$$\frac{d^2y}{dx^2} \cdot \frac{d^2x}{dt^2} \neq \frac{d^2y}{dt^2}.$$

In fact, the notation itself, by the placing of the indices 2, carries a warning against such an inference.

EXERCISES

1. Simplify by inspection:

$$\frac{dQ}{dx} \cdot \frac{dx}{dt}, \qquad \frac{dV}{dr} \cdot \frac{dr}{dt}, \qquad \frac{dy}{dt} \div \frac{dx}{dt}, \qquad 1 \div \frac{dy}{dx}.$$

2. Write the differential of S, if $S = 4\pi r^2$. Divide by dt to find dS/dt. Similarly for dS/dp and dS/dr.

3. Express in derivative notation each of these statements:

$$dy = x^2 dx, \qquad dS = \frac{7}{x}\,dx, \qquad du = \frac{dt}{t^2}, \qquad dR = s\,ds.$$

4. Write the differential of y, if $y = x^4$. Approximately what change in y if x increases from 2 to 2.004? From 1.999 to 2.001?

5. Write the differentials of the following:

(a) $\dfrac{\sqrt{x^2+81}}{7} + \dfrac{5}{10-x}$, (b) $3^{-\left(\frac{x}{a}\right)^2}$, (c) $\log \dfrac{x^2}{\sqrt{x^4+1}}$,

(d) $\log(\csc\theta + \operatorname{ctn}\theta)$, (e) $6\tan\dfrac{\theta}{2}$, (f) $a\sec^5 .2\,\theta$,

(g) $\operatorname{vers}^{-1}(1-\sqrt{1-x^2})$, (h) $\operatorname{ctn}^{-1} x^2$, (i) $(\sec^{-1} 4\,x)^2$.

6. The following formulas relate to various physical problems. In each case express the approximate change in the first-named quantity, which would result from a slight change in the variable indicated in brackets.

(a) Period of Condenser Discharge: $T = 2\pi\sqrt{LS}$, [S].

(b) Heat developed by Electric Current: $H = \dfrac{RC^2t}{4.19 \times 10^7}$, [$C$].

(c) Torsion of a Twisted Wire: $\phi = \dfrac{2\,lu}{n\pi r^4}$, [$r$].

(d) Flow through a Fine Tube: $V = \dfrac{\pi p r^4}{8\,l\eta}$, [$l$].

(*e*) Period of a Vibrating Magnet: $T = 2\pi\sqrt{\dfrac{K}{MH}}$, [$H$].

(*f*) Current through a Galvanometer: $i = \dfrac{Hr}{2\pi n}\tan\theta$, [$\theta$].

7. The average power for a certain electric circuit is $P = 440\cos\phi$. About what change in P if ϕ increases from $.2^{(r)}$ to $.204^{(r)}$?

8. Approximately what change is there in $\log_{10} x$ while x increases from 5 to 5.000002? If $\log_{10} 5 = .69897\ 00043$, find $\log_{10} 5.000002$.

9. Given $\sin 59°\ 12' = .85895\ 98978$, find $\sin 59°\ 12'.003$, approx.

10. Given $e^{2.5} = 12.182494$, find $e^{2.5002}$, approx.

11. Approximately what error would there be in the calculated area and volume of a sphere if the diameter were measured as 10.02 in. when in reality 9.98 in.?

12. The rate of rotation (R deg./hr.) of a line on the earth's surface relative to the plane of a Foucault pendulum, in latitude L, is $R = 15\sin L$. Approximately how much greater is R in latitude 45°.002 than in latitude 45°?

13. In a railway curve of radius 2000 ft. a section of track is to subtend a central angle of 2°. How far will it be from its chord at the middle? About how much farther if the angle were 2°.008?

14. The pull (P lb.) exerted by a man's biceps in lifting a 20 lb. weight in a certain position was

$$P = 24\sqrt{26 - \cos\phi}\,\frac{\cos(120° - \phi)}{\sin\phi},$$

where ϕ was the angle between the bones of forearm and upper arm. Find the approximate change in P while ϕ changed from 90° to 89°.99.

15. Where and at what inclination angles do the following curves cut or meet the X- and Y-axes?

(*a*) $y = x^2 - 7x + 10$, (*b*) $y = (x-2)\sqrt{x-5}$, (*c*) $y = \sqrt{x^2 - 5x}$,

(*d*) $\dfrac{x}{2} + \dfrac{y}{3} = 1$, (*e*) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, (*f*) $\dfrac{x^3}{a^3} + \dfrac{y^3}{b^3} = 1$,

(*g*) $\left(\dfrac{x}{a}\right)^{\frac{1}{2}} + \left(\dfrac{y}{b}\right)^{\frac{1}{2}} = 1$, (*h*) $y = \tan x$, (*i*) $y = (x-1)^2(x-5)^{\frac{1}{3}}$.

PART III. PARTIAL DERIVATIVES

§ 43. The Idea. A quantity Q may depend upon several variables x, y, u, v, etc., each of which may change independently of the others. Often we need to single out the

effect of changing one variable alone, *e.g.*, to find how fast Q will change with x, if y, u, v, etc., all remain fixed.

As always in finding an instantaneous rate, we differentiate. But here we treat y, u, v, etc., as constants. The result is called the *partial derivative* of Q with respect to x, written $\partial Q/\partial x$.

Ex. I. If $Q=x^3-7\,x^2y+3\,xu^2+6\,y^2u-10$,

then $$\frac{\partial Q}{\partial x}=3\,x^2-14\,xy+3\,u^2.$$

Likewise $$\frac{\partial Q}{\partial y}=-\,7\,x^2+12\,yu; \qquad \text{etc.}$$

Ex. II. The temperature (T deg.) at any point in the earth's crust varies with the latitude ($L°$), the depth (D ft.) below the surface, the local time of day (t hr.), the age of the earth (A yr.), etc. Interpret $\partial T/\partial L$, $\partial T/\partial D$, etc.

Here $\partial T/\partial L$ gives the rate of change of the temperature per degree change in latitude at some one depth and at some one instant. Similarly $\partial T/\partial D$ gives the rate at which T changes with the depth alone, "other things being equal." And so on.

§ 44. **Geometric Interpretation.** Any equation which gives one variable z as a real function of two others, x and y, may be regarded as the equation of some surface. Simply let z be the height of the surface above any point (x, y) of a horizontal base-plane: the equation tells how the height varies from point to point.

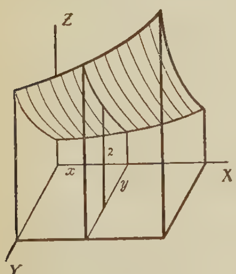


FIG. 28.

To plot the surface we calculate from the equation a table of heights z for various base-points (x, y), erect the proper ordinates, and join. (Cf. *Intro.*, § 296 ff.)

Note the positions of the positive X- and Y-axes in Fig. 28.

If a surface be cut by any vertical plane in which x is constant, z varies with y alone along the curve of intersection; and $\partial z/\partial y$ is the slope of the section. Likewise $\partial z/\partial x$ is the slope of any section in which y is constant.

Ex. I. The slope at $x=2$ of the section cut from the surface $z=x^3-2\,xy+3\,y^2$ by the plane $y=3$ is

$$\frac{\partial z}{\partial x}=3\,x^2-2\,y=12-6=6.$$

There is no similar geometrical representation for a quantity Q which is a function of three or more independent variables, — at least, in the ordinary three-dimensional space of our perceptions.

§ **45. Higher Derivatives.** The result of differentiating $\partial z/\partial x$ again partially with respect to x is denoted by $\partial^2 z/\partial x^2$. In Ex. I of § 44 it would give the rate at which the slope changes along the section, *i.e.*, the flexion at any point.

If we differentiate z twice with respect to x regarding all other variables as constants, and then differentiate the result once with respect to y regarding x, etc., as constants, the final result is denoted by

$$\frac{\partial^3 z}{\partial y \partial x^2}\cdot \quad \text{(Note the order.)}$$

Similar notations are used for other higher derivatives.

The order in which any successive differentiations are performed is immaterial in most cases, — always, indeed, when the successive results are continuous functions of all the variables concerned.*

E.g., if $$z=x^5+x^4y^2-y^4-20\,x+30\,y,$$
then for one order:

$$\frac{\partial z}{\partial x}=5\,x^4+4\,x^3y^2-20, \quad \text{and } \frac{\partial^2 z}{\partial y \partial x}=8\,x^3y;$$

while for the other order:

$$\frac{\partial z}{\partial y}=2\,x^4y-4\,y^3+30, \quad \text{and } \frac{\partial^2 z}{\partial x \partial y}=8\,x^3y.$$

* Cf. Goursat-Hedrick: *Mathematical Analysis*, v. 1, p. 13.

§ 46. Caution. Some properties of ordinary derivatives do not hold for partial derivatives in general.

Consider, for instance, the consistent equations:

$$x=r\cos\theta,\qquad y=r\sin\theta,\qquad r^2=x^2+y^2. \tag{49}$$

The first and last give

$$\frac{\partial x}{\partial r}=\cos\theta,\qquad \frac{\partial r}{\partial x}=\frac{x}{r}\ (=\cos\theta).$$

Here it is not true that $\dfrac{\partial x}{\partial r}\cdot\dfrac{\partial r}{\partial x}=1.$

The reason is that in getting $\partial x/\partial r$ we held θ constant; while in getting $\partial r/\partial x$ we held y constant.

Again, from the gas law, $pv=kT$:

$$\frac{\partial p}{\partial T}=\frac{k}{v},\qquad \frac{\partial T}{\partial v}=\frac{p}{k},\qquad \frac{\partial p}{\partial v}=-\frac{kT}{v^2}.$$

Thus $\dfrac{\partial p}{\partial T}\cdot\dfrac{\partial T}{\partial v}=\dfrac{p}{v}$, while $\dfrac{\partial p}{\partial v}=-\dfrac{p}{v}$.

Hence the product $\dfrac{\partial p}{\partial T}\cdot\dfrac{\partial T}{\partial v}$ does not equal $\dfrac{\partial p}{\partial v}$.

Why does the usual rule fail? What variable was held constant in each differentiation?

EXERCISES

1. $Q=x^2y-y^3+xz^2+xyz$. Find $\dfrac{\partial Q}{\partial y}$ and $\dfrac{\partial Q}{\partial z}$.

2. $z=x^2\log y+y\sin(x+y)$. Find $\dfrac{\partial^2 z}{\partial y^2}$.

3. $u=\tan^{-1}xy-e^y$. Verify that $\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$.

4. $z=x^6+\dfrac{x}{y^2}-10\,y^4$. Verify that $\dfrac{\partial^3 z}{\partial x^2\partial y}=\dfrac{\partial^3 z}{\partial x\partial y\partial x}$.

5. Plot the surface $z=x^2+y^3$ above that part of the XY-plane extending from $x=0$ to $x=3$ and from $y=0$ to $y=4$. What is the nature of the section made by any plane in which y is constant?

6. In Ex. 5 find the slope and flexion at $y=2$ of the section made by the plane $x=3$.

7. Find the slope and flexion at $x=3$ of the section cut from the surface $z=x^2-10y+100$ by any plane $y=k$.

8. From equations (49), § 46, find $\partial y/\partial r$ and $\partial r/\partial y$. See whether these are reciprocals. Explain.

9. In the gas law: $pv=kT$, see whether $\dfrac{\partial v}{\partial T}\cdot\dfrac{\partial T}{\partial p}=\dfrac{\partial v}{\partial p}$. Explain.

10. In Ex. 9, show that $\dfrac{\partial v}{\partial p}\cdot\dfrac{\partial p}{\partial v}=1$. Explain.

11. In the following, state the meaning of the partial derivative of the function with respect to each independent variable in question, using letters as needed:

(*a*) The strength of a steel beam, *i.e.*, the load it can just carry, depends upon its length, width, and thickness, the percentage of carbon in the steel, etc.

(*b*) The rate at which a plant grows depends upon the amount of fertilizer added to the soil, the solar energy and moisture received, its age, etc.

(*c*) The speed of a particle in a jet of water issuing from a nozzle varies with the distance from the nozzle, the distance from the middle of the jet, the pressure in the hose, etc.

(*d*) The dividend on a life insurance policy depends upon the percentage of expected mortality realized, the mean rate of interest earned, the saving on expenses of management, and the age of the policy.

(*e*) An illustration of your own finding.

12. The market price of a secure 4% $1000 bond depends upon the time it has to run and upon the present market rate of interest. How could P be plotted as a function of T and r? What geometrical meaning would $\partial P/\partial T$ have? What investment meaning?

13. The temperature of a rectangular plate at any point x inches from one end and y inches from one side was $T=200+20x+10y-x^2-y^2$. Find the *temperature gradient* in the X-direction at (8, 6), likewise in the Y-direction. (The temperature gradient is the rate at which T changes per inch.)

14. The density of a thin rectangular plate at any point (x, y) is $D=20+8x+12y-x^2-2y^2$. Find the density gradients in the X- and Y-directions at (4, 3).

§ **47. Total Derivative.** Suppose now that all the independent variables, x, y, etc., change at once, instead of some

remaining fixed. Their rates of change, dx/dt, dy/dt, etc., may have any values, quite independently. Then, as we shall see below, the rate at which the function z changes is:

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}+\cdots. \tag{50}$$

That is, the total derivative of z equals the sum of the several partial derivatives of z, each multiplied by the derivative of the corresponding independent variable.

To illustrate, suppose that

$$z=x^4+2\,x^3y+6\,y^2u^2.$$

Then $\dfrac{dz}{dt}=(4\,x^3+6\,x^2y)\dfrac{dx}{dt}+(2\,x^3+12\,yu^2)\dfrac{dy}{dt}+12\,y^2u\dfrac{du}{dt}.$

This result could also be obtained without (50) by simply differentiating as formerly for powers and products and collecting results.

Formula (50) is not necessary for routine differentiation. It is, however, very useful in deriving further principles, and also in rate problems where no formula is given but the values of $\partial z/\partial x$ and $\partial z/\partial y$ are known, together with dx/dt and dy/dt.

Derivation of formula (50)

Let us now see how (50) is derived. At any instant,

$$z=f(x,\,y,\,\cdots).$$
$$\therefore\quad z+\Delta z=f(x+\Delta x,\,y+\Delta y,\,\cdots).$$

The average rate of change of z per unit of time is then:

$$\frac{\Delta z}{\Delta t}=\frac{f(x+\Delta x,\,y+\Delta y,\,\cdots)-f(x,\,y)}{\Delta t}. \tag{51}$$

To isolate the effect of changes separately in x, y, etc., we subtract and add like terms in the numerator of (51),

so as to get fractions in each of which only one variable changes.

$$\frac{\Delta z}{\Delta t}=\frac{f(x+\Delta x,\ y+\Delta y,\ \cdots)-f(x,\ y+\Delta y,\ \cdots)}{\Delta x}\frac{\Delta x}{\Delta t}$$
$$+\frac{f(x,\ y+\Delta y,\ \cdots)-f(x,\ y,\ \cdots)}{\Delta y}\frac{\Delta y}{\Delta t}\cdots. \tag{52}$$

E.g., in the first fraction, x only is different in the two parts.

Proceeding to the limit, letting Δz, Δx, Δy, etc., all approach zero with Δt, we arrive at equation (50).

Certain critical questions arise in some of these steps which will not be discussed here.*

§ 48. Partial and Total Differentials. The product

$$\frac{\partial z}{\partial x}dx, \quad \text{denoted briefly by } \partial_x z,$$

is called the x-partial differential of z. It is the approximate amount of change in z while x increases by dx, and the other variables remain fixed. Exactly, it is the amount z would change if the instantaneous rate $\partial z/\partial x$ were maintained throughout the interval.

Likewise, $\frac{\partial z}{\partial y}dy$, or $\partial_y z$, is called the y-partial differential of z, with a similar meaning.

From equation (50), on multiplying by dt:

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy\ \cdots,=\partial_x z+\partial_y z\ \cdots \tag{53}$$

That is, the total differential of z equals the sum of the several partial differentials of z.

Or, the total change in z due to slight changes in all the independent variables is approximately the sum of the several changes in z due to the changes in x, y, etc., separately.

* Cf. Goursat-Hedrick: *Mathematical Analysis*, v. 1, pp. 15–16.

EXERCISES

1. In each of the following write the total derivative of z with respect to t, using formula (50), p. 70. Also find it without formula (50) and check.

(a) $z=x^3y^2+x^2y^4+2\,xy$, (b) $z=x^2\log y+e^x\cos y$,

(c) $z=\cos x\cos y+\sin x\sin y$, (d) $z=x\tan^{-1}\dfrac{y}{x}$.

2. Write the total differential of each of the following:

(a) $z=\dfrac{10\,x}{y}$, (b) $w=\sec x\sec y$.

(c) $X=r\cos\theta$, (d) $Q=y\sin^{-1}x+\dfrac{\sqrt{1-x^2}}{y}$.

3. Given $z=e^x\sin y$, $dx/dt=.3$, and $dy/dt=.2$; find dz/dt at the instant when $x=1$ and $y=\dfrac{\pi^{(r)}}{3}$.

4. At a certain instant: $\partial Q/\partial x=8$, $\partial Q/\partial y=5$, $\partial Q/\partial z=3$, x is increasing at the rate of .02 per sec., y at the rate of .04 per sec., and z decreasing at the rate of .15 per sec. How fast was Q then changing?

5. If $z=x^3y^2+7$, find the approximate change in z if x increases from 4 to 4.002 while $y=5$. Also the approximate change in z if y simultaneously changes from 5 to 5.001.

6. An electric current C, for any electromotive force E and resistance R, is $C=E/R$. Find the approximate change in C if R is increased from 8 to 8.01, E being 20. Also the change in C if E is simultaneously decreased from 20 to 19.98.

7. The frequency of vibration of a string of a musical instrument for its fundamental tone is

$$n=\frac{1}{2\,rl}\sqrt{\frac{T}{\pi d}},$$

where l is the length, r the radius and d the density of the string, and T the tension to which it is subjected. Express by a formula the approximate effect of increasing l by a slight amount. Or T by a slight amount. Or both simultaneously.

8. The time of vibration of a magnet, of magnetic moment M and moment of inertia K, in a field of strength H is

$$T=2\,\pi\sqrt{\frac{K}{MH}}.$$

Express by a formula the approximate total change in T due to slight changes in both M and H.

9. For a diffraction grating with rulings at a distance s apart, the wave length L for the spectrum of first order is

$$L = s(\sin I + \sin D),$$

where I and D are the angles of incidence and diffraction. Express by a formula the approximate change in L due to small changes dI and dD in the two angles.

10. For a radio condenser having two co-axial cylinders of radii R and r and length L the capacity for any dielectric constant K is

$$C = K \frac{.2416\, L}{\operatorname{Log} R/r}, \quad \text{base } 10.$$

Express the approximate change in C corresponding to small changes in L, R, and r if made separately or simultaneously.

11. An equation from which the time can be found by observation of the sun is

$$\cos z = \sin \delta \sin \phi + \cos \delta \cos \phi \cos t,$$

where z is the sun's zenith distance, δ its declination, ϕ the latitude of the observer, and t the hour angle. Express the error dt produced by a small error in the measured zenith distance; likewise the error dt produced by a small error $d\phi$ in the assumed latitude. (Keep δ constant in both cases.)

12. One of the formulas used in computing the time of a solar eclipse is

$$X = \rho \cos \phi \sin (\mu - \lambda).$$

Given $\rho = 3953$, $\phi = 60°$, $\lambda = 120°$; also, at a certain instant, $\mu = 125°$ and μ is decreasing at the rate of .25° per min. Approximately what is the change in X during the next minute?

§ 49. Exact Differentials.

An expression of the form

$$M\,dx + N\,dy, \tag{54}$$

where M and N are functions of two independent variables x and y, may or may not be the differential of some function $z = f(x, y)$.

E.g., $(2x + y \cos x)dx + (\sin x + 10)dy$ is the total differential of

$$x^2 + y \sin x + 10y,$$

as we can verify by differentiating the latter quantity.

But $(2x+y\sin x)\,dx+(\cos x+10)dy$ can not be the differential of any function at all if x and y are independent, — as the criterion below will show.

If (54) is to be the exact differential of some function z, the coefficients of dx and dy must be

$$M=\frac{\partial z}{\partial x}, \qquad N=\frac{\partial z}{\partial y}.$$

Differentiating M partially with respect to y, and N with respect to x, would give $\partial^2 z/\partial y\partial x$ and $\partial^2 z/\partial x\partial y$, respectively. Under ordinary conditions these are equal. (§ 45.)

Hence, for (54) to be an exact differential, it is necessary that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}. \tag{55}$$

Conversely, it may be shown that whenever (55) is fulfilled, (54) is sure to be an exact differential. (Cf. § 210.)

For an expression $Mdx+Ndy+Pdu\cdots$ to be exact in several independent variables, the criterion is like (55), for every pair of coefficients. Thus

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial u}=\frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial u}=\frac{\partial P}{\partial y}, \cdots \tag{56}$$

These criteria (55) and (56) are particularly important in thermodynamics. There an expression is often known from theoretical considerations to be an exact differential; and, by using (55) or (56), some important relationships may be discovered.

Ex. I. Test $(2x+y\sin x)dx+(\cos x+10)dy$.

Here $M=2x+y\sin x$, $\qquad N=\cos x+10$.

$$\therefore \quad \frac{\partial M}{\partial y}=\sin x, \qquad \frac{\partial N}{\partial x}=-\sin x.$$

The expression is not an exact differential. (See above.)

§ 50. Implicit Functions. The relation between two variables x and y is often given by an equation which is not readily solvable for either variable in terms of the other. In such cases we can find dy/dx as in § 30; but a quicker method is to use a formula now to be derived.

Symbolically expressed, let the equation be

$$f(x, y) = 0. \tag{57}$$

Call the function z temporarily. Then since z is constant, viz. zero, its total differential is zero:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0. \tag{58}$$

Solving for dy/dx gives, if $\partial f/\partial y \neq 0$:

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}. \tag{59}$$

An easy way to remember the order in (59) is to observe that if $\partial f/\partial x$ and $\partial f/\partial y$ were fractions, carrying out the division by inverting the denominator and canceling would reduce the right member of (59) to $-\partial y/\partial x$.

By using (59) we can write at sight the derivative of an implicit function, or the slope of a curve whose equation is given in the implicit form (57). Also we can discuss various theoretical questions readily.

When three or more variables are involved, an equation similar to (59) gives $\partial y/\partial x$ or the partial derivative of any variable with respect to any other.

Ex. I. Find the slope of the curve $x^4 + y^4 + x^3y - 3\,xy^3 = 0$ at the point (1, 1).

Using (59) we write by inspection:

$$\frac{dy}{dx} = -\frac{4\,x^3 + 3\,x^2y - 3\,y^3}{4\,y^3 + x^3 - 9\,xy^2}. \tag{60}$$

Substituting $x = 1$ and $y = 1$ gives $l = 1$, the required slope.

EXERCISES

1. Test whether each of the following is an exact differential:

(*a*) $(2x \log y + y)\, dx + \frac{x}{y}(x+y)dy$,

(*b*) $(\cos y + y \cos x)dx + (x \sin y + \sin x)dy$,

(*c*) $(2xy + z^4)dx + (x^2 + 3y^2 z)dy + (y^3 + 4z^3 x)dz$.

2. Given that $Md\theta + e^{2\theta}(3 \sin 3\theta + 2 \cos 3\theta) \cos \phi d\phi$ is an exact differential, show that $\partial M/\partial \phi = 13 e^{2\theta} \cos 3\theta \cos \phi$.

3. Given that $CdT + (l-p)dv$ is an exact differential, where C, l, p, are variables and $l = T\partial p/\partial T$, show that

$$\frac{\partial C}{\partial v} = T \frac{\partial^2 p}{\partial T^2}.$$

[The calculations in this and the next two exercises are used in certain proofs in thermodynamics.]

4. Given that $SdT - Ede + Vdp$ is an exact differential, show that $-\partial E/\partial p = \partial V/\partial e$. Also deduce two other similar relations.

5. Given that $\left(T\frac{\partial S}{\partial x} - p\frac{\partial v}{\partial x}\right)dx + \left(T\frac{\partial S}{\partial y} - p\frac{\partial v}{\partial y}\right)dy$ is an exact differential, show that

$$\frac{\partial T}{\partial x}\frac{\partial S}{\partial y} - \frac{\partial T}{\partial y}\frac{\partial S}{\partial x} = \frac{\partial p}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial v}{\partial x}.$$

6. In each of the following cases find dy/dx by (59), and check by the method of § 30:

(*a*) $x^2y^3 + x^3y^2 + 2x + 4y = 7$,
(*b*) $x + y \sin x + \log y = 0$,
(*c*) $\tan x \sec y + \log x - e^y = 0$,
(*d*) $\sin^{-1} x - \tan^{-1} y - xy = 0$,
(*e*) $\frac{\sqrt{x^2-1}}{y} + \sec^{-1} x + \operatorname{vers}^{-1} y = 0$.

7. If $x^2 + y^2 + z^2 = 100$, find $\partial z/\partial x$ after first solving for z. Also find $\partial z/\partial x$ implicitly, regarding z and x as the only variables.

8. If $(x^2 + y^2 + z^2)^3 - a^3xyz = 0$, find $\partial z/\partial x$.

9. Find the slope of the curve $x^3 + y^3 - 18xy = 0$ at (4, 8). Also at a point where $x = y$, other than the origin.

10. Find the slope of each of the following curves at an intersection with the X-axis:

(*a*) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$,
(*b*) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$,
(*c*) $x^2 + y^2 = a^2$,
(*d*) $2x^2 + 3y^2 - 5x + y + 3 = 0$.

11. If $2x^2 + y^2 = ke^{-\frac{x^2}{2x^2+y^2}}$, find dy/dx and show that it can be reduced to $-x(4x^2 + 3y^2) \div y(x^2 + y^2)$.

§ 51. Directional Gradient. The temperature T of a hot object at any one instant may vary from point to point. Its gradient or rate of change per unit distance, starting from any one point, is usually different in different directions.

In a thin flat plate T varies with the distances x and y from chosen axes. Here $\partial T/\partial x$ is the gradient in the X-direction; and $\partial T/\partial y$ in the Y-direction. Let s be the distance measured along any other direction, starting from the given point (x_1, y_1). Then dT/ds is the gradient along that direction. By (50), § 47:

$$\frac{dT}{ds}=\frac{\partial T}{\partial x}\frac{dx}{ds}+\frac{\partial T}{\partial y}\frac{dy}{ds}. \qquad (61)$$

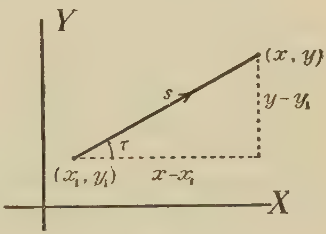


FIG. 29.

But $x-x_1=s\cos\tau$, where τ is the direction angle.
Hence $dx/ds=\cos\tau$. Likewise $dy/ds=\sin\tau$; and (61) becomes

$$\frac{dT}{ds}=\frac{\partial T}{\partial x}\cos\tau+\frac{\partial T}{\partial y}\sin\tau. \qquad (62)$$

Maximum gradient. At any given point $\partial T/\partial x$ and $\partial T/\partial y$ are constants. For what direction angle τ will the gradient in (62) be a maximum? Differentiating, we put

$$\frac{\partial T}{\partial x}(-\sin\tau)+\frac{\partial T}{\partial y}(\cos\tau)=0. \qquad (63)$$

Combining this with $\sin^2\tau+\cos^2\tau=1$, and reducing:*

$$\sin\tau=\pm\frac{\dfrac{\partial T}{\partial y}}{\sqrt{\left(\dfrac{\partial T}{\partial x}\right)^2+\left(\dfrac{\partial T}{\partial y}\right)^2}},\quad \cos\tau=\pm\frac{\dfrac{\partial T}{\partial x}}{\sqrt{\left(\dfrac{\partial T}{\partial x}\right)^2+\left(\dfrac{\partial T}{\partial y}\right)^2}}. \qquad (64)$$

*It is best not to use (64) as formulas. Simply write (62) with numerical coefficients, and maximize as an elementary function of τ.

Substituting these values for $\sin\tau$ and $\cos\tau$ in (62) and simplifying, we find as the maximum value of dT/ds:

$$\textbf{Max. Gradient} = \sqrt{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2}. \tag{65}$$

The − sign would give the minimum gradient, algebraically considered. That is, T would be decreasing most rapidly.

All these equations (61)–(65) hold true not only for the varying temperature of an object but also for any quantity which varies in a definite way from point to point in a given plane. Some illustrations are the strength of a non-uniform magnetic field, the velocity of flowing water, the density of a solid, the height of a hill above its base, etc.

§ 52. Slope of Any Vertical Section of a Surface. Geometrically $\partial z/\partial x$ is the slope of a section of a surface made by a plane $y = c$. (§ 44.) In any other vertical section, the height z varies with the horizontal distance s. (Fig. 30.) The slope dz/ds is a directional gradient; and by (62),

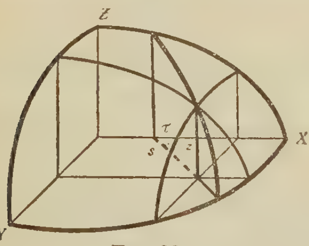


FIG. 30.

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\tau + \frac{\partial z}{\partial y}\sin\tau, \tag{66}$$

where τ is the angle which the vertical cutting plane makes with the X-axis.

The maximum slope for any vertical section through a given point is by (65):

$$\text{Max. slope} = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}. \tag{67}$$

This maximum slope is also called the "slope of the surface" at the point.

§ 53. Slope and Inclination of a Tangent Plane. The inclination γ (Greek letter *gamma*) of any plane BAP

(Fig. 31) is the angle HAP formed by two lines, AP in the given plane and AH in a horizontal plane, both perpendicular to the intersection line AB. The slope of plane ABP is defined as the amount of rise of AP per horizontal unit along AH. Hence

$$l = \frac{HP}{AH} = \tan\gamma. \tag{68}$$

This is also the slope of the *line* AP.

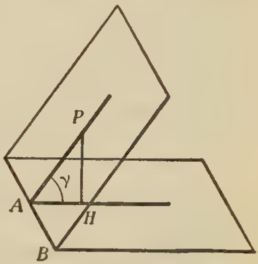


FIG. 31.

Evidently AP is the steepest line in plane BAP through A. If this plane is tangent to a surface at A, line AP will be tangent to the steepest vertical section of the surface through A. Hence the slope l of the tangent plane is the maximum sectional slope; or, by (67):

$$l = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \tan\gamma. \tag{69}$$

Both l and the inclination γ can be found from (69).

For some later purposes we shall need to know also $\sec\gamma$. Since $\sec^2\gamma = 1 + \tan^2\gamma$,

$$\sec\gamma = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}. \tag{70}$$

EXERCISES

1. The temperature ($T°$) at any point (x, y) in a flat plate is $T = 500 + 10\,x + 6\,y - x^2 - y^2$. What is the temperature gradient at (3, 4) in the positive X-direction? Y-direction? In the direction $\tau = 60°$? $\tau = 315°$? What is the maximum gradient at (3, 4), and for what value of τ?

2. The surface density of a plate varies thus: $D = 12 + .03\,x^2 + .02\,y^2$. Find the density gradient at (10, 10) in the direction $\tau = 135°$. Also find the maximum gradient at (10, 10), and what τ gives it.

3. The coefficient of friction of a flat plate varies thus: $f = .01[10+4x+6y-x^2-y^2]$. Find the maximum gradient of f at (3, 4), and what τ gives it.

4. The brightness of illumination of a rectangular garden varies thus with the distance r ft. from one corner: $I=3000/(900+r^2)$. Find the maximum gradient of I at (20, 10). The coördinate axes originate at the corner mentioned.

5. Find the slope of a vertical section of the surface $z=x^2+y$ through (2, 1, 5) made by a plane for which $\tau=30°$. Also find the slope of the steepest section through that point.

6. The same as Ex. 5 for these surfaces and points:

(*a*) $z=4x^2+y^2$, at (1, 2, 8); (*b*) $x^2+y^2+z^2=169$, at (3, 4, 12);
(*c*) $z=2x^2-y^2$, at (3, 3, 9); (*d*) $2x+3y+4z=36$, anywhere.

7. Find the slope and inclination of the plane tangent to the surface $z=x^2+y^2$ at (1, 1, 2).

8. The same as Ex. 7 for these surfaces and points:

(*a*) $10z=2x^2+y^2$, at (1, 3, 1.1); (*b*) $z=x^2+y^2$, at (1, −2, 5);
(*c*) $x^2+y^2+z^2=4$, at (1, 1, $\sqrt{2}$); (*d*) $z=xy$, at (3, 3, 9);
(*e*) $4x^2+z^2=8$, at (1, 5, −2); (*f*) $z=x^3+y^2$, at ($\frac{1}{2}$, 1, $\frac{9}{8}$).

9. Find the slope and inclination of the plane $z=x+3y+12$. What direction τ gives the steepest lines in the plane?

10. The same as Ex. 9 for the planes

(*a*) $z=20-2x+y$, (*b*) $4x+3y+10z=60$.

11. Check (70) by the homogeneity test.

12. Derive (64) from (63) in detail. Also derive (65).

§ 54. Maxima and Minima in General. Let z be a function of several independent variables x, y, u, etc.; and let z reach a maximum for the set of values: $x=x_1$, $y=y_1$, $u=u_1$, etc. Then, unless z rises sharply to its maximum, we must have there:

$$\frac{\partial z}{\partial x}=0, \qquad \frac{\partial z}{\partial y}=0, \qquad \frac{\partial z}{\partial u}=0, \text{ etc.} \tag{71}$$

For, the maximum value must be the greatest obtainable in the vicinity by varying x alone and keeping y, u, etc., constant. Likewise by varying y alone; etc.

In case there are only two independent variables x and y, we may also state the matter thus: Regard z as the height of a surface. At a summit point, the tangent plane is horizontal and its slope is zero. By (69) this requires both $\partial z/\partial x = 0$ and $\partial z/\partial y = 0$.

Testing the sign of $\partial z/\partial x$ for values of x before and after x_1 (with $y = y_1$, etc.), — and similarly for $\partial z/\partial y$, etc., — often helps us to see whether we are getting a maximum or a minimum, or neither. In many practical problems no test is needed. Tests sufficient to remove all doubt are sometimes rather complicated.*

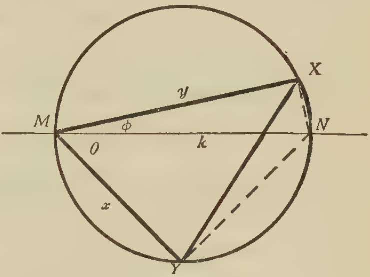


FIG. 32.

Before applying equations (71) we must be sure that x, y, u, etc., are really independent. *We can utilize any given conditions to eliminate superfluous variables.*

Ex. I. Prove that the largest triangle which can be inscribed in a circle is equilateral.

Let two sides MX and MY make angles ϕ and θ with the diameter MN. Taking x or y as base, the area is easily found to be

$$A = \tfrac{1}{2}\, xy \sin(\theta + \phi). \tag{72}$$

It would be incorrect to apply (71) here and put $\partial A/\partial x = 0$, etc.; for x, y, ϕ, θ, are not all independent.

From $\triangle MNY$ and $\triangle MNX$, $x = k\cos\theta$, $\quad y = k\cos\phi$.

$$\therefore \qquad A = \tfrac{1}{2}\, k^2 \cos\theta \cos\phi \sin(\theta + \phi). \tag{73}$$

Now θ and ϕ can be changed independently. Hence by (71):

$$\frac{\partial A}{\partial \theta} = \frac{1}{2} k^2 \cos\phi\,[\cos\theta \cos(\theta+\phi) - \sin\theta \sin(\theta+\phi)] = 0,$$

* Cf. Goursat-Hedrick: *Mathematical Analysis*, v. 1, pp. 118–127.

with a similar equation for $\partial A/\partial\phi$. Recalling that $\cos u \cos v - \sin u \sin v = \cos(u+v)$, we may write

$$\frac{\partial A}{\partial\theta}=\frac{1}{2}k^2\cos\phi\cos(2\theta+\phi)=0, \tag{74}$$

$$\frac{\partial A}{\partial\phi}=\frac{1}{2}k^2\cos\theta\cos(2\phi+\theta)=0. \tag{75}$$

From (74), either $\cos\phi=0$, which gives $y=0$ and $A=0$; or else $\cos(2\theta+\phi)=0$, which gives $2\theta+\phi=90°$. Likewise for (75).

$$\therefore\ 2\theta+\phi=90°, \qquad \text{and } 2\phi+\theta=90°.$$

Hence we find easily that $\theta=\phi=30°$. Thus $\theta+\phi=60°$; and the triangle must be equilateral. (*Q. E. D.*)

EXERCISES

1. Test for a maximum or minimum value:

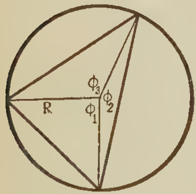
(*a*) $z=x^2+y^2-6x-4y+50$, (*b*) $z=x^2-y^2-10x+50$,

(*c*) $Q=10x+6y+4z-x^2-y^2-z^2$, (*d*) $S=xy+\dfrac{80}{x}+\dfrac{100}{y}$.

2. The surface density of a flat plate varies thus: $D=5+.04x+.06y-.01x^2-.01y^2$. Find the maximum D.

3. An uncovered rectangular tank is to contain 12000 cu. ft. The cost per sq. ft. for the base will be 40¢, for the front and back 80¢, and for the sides 60¢. For what dimensions will the total cost be least?

4. A firm is to use several million wooden boxes, each to contain 8000 cu. in. The lumber for the ends is to be twice as thick as for the top and bottom, and four times as thick as for the sides. Ignoring the overlap where nailed together, what are the most economical dimensions?



5. Solve Ex. I of § 54 also as follows: Let ϕ_1, ϕ_2, ϕ_3 be the angles at the center subtended by the sides of the triangle, and R the radius of the circle. Show that $A=\frac{1}{2}R^2(\sin\phi_1+\sin\phi_2+\sin\phi_3)$; also that $\sin\phi_3=-\sin(\phi_1+\phi_2)$. Then take such further steps as are needed.

6. Two factories make a certain implement. One produces each month a number x at an expense of $\$E_1$; the other a number y at an expense of $\$E_2$, where

$$E_1=50000+12x+.001x^2, \qquad E_2=30000+6y+.002y^2.$$

The price (\$$p$ each) at which the total output can just be sold is $p = 50 - .001\,x - .001\,y$. (*A*) Express the separate monthly profits \$$P_1$ and \$$P_2$. (*B*) If each factory tries to maximize its own profit, regarding the competitor's output as independent of its own, find x and y then. Also find the corresponding price and the profits obtained.

7. In Ex. 6 if the two factories come under joint ownership which regulates x and y so as to make the *combined* profits a maximum, find x, y, p, and the total profit.

8. Because of slight inaccuracies Table I will not quite satisfy a linear formula, $y = a + bx$. Find what values of a and b will make the formula fit the table most closely. [Hint: When $x = 2$, the formula requires $y = a + 2\,b$ while the table gives $y = 12.8$. The difference $a + 2\,b - 12.8$ is an error. Likewise when $x = 4$, the error is $a + 4\,b - 19.1$. The best values for a and b are those which make *the sum of the squares of all the errors least.* (Cf. *Intro.*, §§ 342–3.)]

TABLE I

x	y
2	12.8
4	19.1
6	25.2
8	30.9

9. The same as Ex. 8 for Table II below.

10. The same as Ex. 8 for Table III and the formula $y = a + bx + cx^2$.

TABLE II

x	10	20	30
y	12.9	10.8	9.3

TABLE III

x	0	1	2	3
y	5.1	7.9	13	20.1

EXERCISES FOR REVIEW

1. Differentiate, and simplify each derivative:

(*a*) $2\,\theta \sin\theta + (2 - \theta^2)\cos\theta$, (*b*) $\log x \cdot \log(\log x) - \log x$,

(*c*) $e^{2x}(\cos 3\,x - \frac{2}{3}\sin 3\,x)$, (*d*) $\operatorname{ctn}^{-1}\frac{x}{4} + \log\sqrt{\frac{x-4}{x+4}}$,

(*e*) $\frac{1}{4}\tan^4\theta - \frac{1}{2}\tan^2\theta + \log\sec\theta$, (*f*) $\log(x - 4 + \sqrt{x^2 - 8\,x + 25})$.

2. Find the second derivative of each of the following:

(*a*) $(2 - 6\,x + 15\,x^2)(1 + 2\,x)^{\frac{3}{2}}$,

(*b*) $(x^3 + 10\,x)\sqrt{x^2+4} + 24\log(x + \sqrt{x^2+4})$,

(*c*) $\dfrac{2}{\sqrt{4\,ac - b^2}}\tan^{-1}\dfrac{2\,cx + b}{\sqrt{4\,ac - b^2}}$,

(*d*) $\dfrac{1}{\sqrt{b^2 - 4\,ac}}\log\dfrac{2\,cx + b - \sqrt{b^2 - 4\,ac}}{2\,cx + b + \sqrt{b^2 - 4\,ac}}$.

3. Write the differentials of

(*a*) $\sin^{-1}\dfrac{x}{a}$, (*b*) $\log\dfrac{x-a}{x+a}$, (*c*) $\log(x+\sqrt{x^2\pm a^2})$,

(*d*) $\tan^{-1}\dfrac{x}{a}$, (*e*) $\sec^{-1}\dfrac{x}{a}$, (*f*) $\text{vers}^{-1}\dfrac{x}{a}$.

4. If $y=u^v$, where u and v are functions of x, find dy/dx logarithmically. Observe that the result could be written as the sum of two parts: one obtained as if v were constant; the other as if u were constant. [Cf. § 47, (50).]

5. Find dy/dx if $y\sin^{-1}x+x\sin^{-1}y=1$.

6. Find dv/dp if $\left(p+\dfrac{a}{v^2}\right)(v-b)=c$.

7. Find the flexion of each of the following curves at any point:

(*a*) $x^2+2xy-y^2=5$, (*b*) $3x^2+7xy+y^2=20$.

8. If $u=(x^2+y^2+z^2)^{-\frac{1}{2}}$, find $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$.

9. The fluctuating temperature of the ground, x cm. below the surface and t sec. after the beginning of the daily cycle, is

$$T=ae^{-kx}\sin(\omega t-kx),$$

where a, k, and ω are positive constants. Find $\partial T/\partial x$ and $\partial T/\partial t$, at $x=0$ and $t=0$. What does each derivative mean, physically?

10. In Ex. 9 express algebraically each of the following:

(*a*) The time t at which T is a maximum, for any depth x;

(*b*) The depth x at which T is a maximum, for any time t.

11. The hydraulic efficiency of a turbine of a certain type is

$$E=\frac{1}{gh}[.94\,v\sqrt{v^2+2\,gh}-v^2],$$

where v is the speed of the water entering the discharge nozzle, and g and h are constants. What v makes E a maximum?

12. At a certain point in a rod, the stress across a plane, of inclination ϕ, was $p=4000-1000\cos 2\phi-500\sin 2\phi$. Find the maximum p, and what ϕ gives it.

13. Test for a maximum or minimum: $z=\dfrac{x^2}{4}-\dfrac{y^2}{9}$.

14. The temperature of a flat plate at any point (x, y) was $T=100+20x+16y-x^2-y^2$. Find the maximum T and where it occurred. Also find the maximum gradient of T at $(5, 10)$; and in what direction it ran.

15. The centrifugal force of the earth's rotation tends to move objects toward the equator. Regarding the earth as a sphere, the acceler-

ation is proportional to $\sin L \cos L$, where L is the latitude. For what L is this greatest?

16. The acceleration in Ex. 15 is replaced by

$$A = \frac{k \sin 2L}{\sqrt{.99672 + .00328 \cos 2L}}$$

if we consider the earth's flattening. What L gives the maximum A?

17. Find the dimensions of the largest rectangle that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$. [Hint: Taking a point (x, y) as one vertex, what are the dimensions?]

18. Solve Ex. 17 without differentiation by using the auxiliary angle ϕ. (§ 16.)

19. (a) The same as Ex. 17 for the curve $4x^{\frac{2}{3}} + 9y^{\frac{2}{3}} = 36$. ($b$) Solve also without differentiation by using a parameter ϕ, as in § 17, p. 27.

20. How high above the center of a circular garden of radius 30 ft. should a light L be hung to illuminate most brightly a narrow walk at the garden's edge? The brightness will vary as $(\sin I)/D^2$ where I is the inclination of the light rays and D is the distance from L to the walk.

21. A certain formula in Physical Chemistry reads thus:

$$k = \frac{1}{t(a+b)} \log \frac{b(a+x)}{a(b-x)}.$$

Find the approximate change in k for a small change in a, x, or t, alone.

22. Meeh's somewhat rough formula for the area (A sq. m.) of an average human body of any weight (W kg.) is $A = .123\sqrt[3]{W^2}$. Approximately what difference in A for $W = 64$ and $W = 64.02$?

23. A better formula than the foregoing is that of Du Bois: $A = 71.84\, W^{.425}\, H^{.725}$, where H cm. is the height. Approximately what change in A if we increase W from 64 to 64.02 and decrease H from 170 to 169.8? [Here A is in sq. cm.]

24. In Ex. 14 about what difference between the temperatures at (7.98, 4.99) and at (8.01, 5.02)?

25. Approximately what change in the volume of a metal block if the width increased from 4 to 4.0006, and the thickness from 2 to 2.0003, while the length was compressed from 200 to 199.9?

CHAPTER III

INTEGRALS

PART I. FIRST PRINCIPLES *

§ 55. Indefinite Integrals. Integration is the reverse of differentiation: viz. finding a function when given its derivative or differential.

If $f(x)dx$ is the differential of some function $F(x)$, then $F(x)$ is called an *integral* of $f(x)dx$, or an integral of $f(x)$ with respect to x, written

$$\int f(x)\,dx.$$

We speak of *an* integral rather than *the* integral because $F(x)+C$ (where C is any constant whatever) is also an integral. We may, however, call $F(x)+C$ *the* integral, since for some value of C this includes every possible integral.

The arbitrary constant C which should be added in integrating is called the "constant of integration." Its value can be determined if we know the value of the required integral for any one value of x. The integral $F(x)+C$ is *indefinite* until C is known.

The expression which is to be integrated is called the *integrand*.

§ 56. Some Basic Formulas. The following integration formulas are established in the *Introduction*. They also follow at once from the corresponding differentiation formulas of § 28 above. A more extended list will be obtained

* Summarized, in part, from *Intro.*, Chap. IV, VII, X, XII.

shortly. Any integration can be checked by differentiating the supposed integral.

$$\int u^n\,du = \frac{u^{n+1}}{n+1}+C, \text{ (if } n\neq -1), \tag{1}$$

$$\int \frac{1}{u}\,du = \log u+C, \text{ (base } e), \tag{2}$$

$$\int e^u\,du = e^u + C, \tag{3}$$

$$\int \sin u\,du = -\cos u+C, \tag{4}$$

$$\int \cos u\,du = \sin u+C. \tag{5}$$

Formulas (4), (5) are valid only for angles in radians. For degrees divide the results by .017453 or multiply by 57.296.

To integrate $a^u\,du$ replace (3) by

$$\int a^u du = \frac{1}{\log a}a^u + C.$$

The integral of an algebraic sum of several terms equals the sum of the integrals of the individual terms. One constant of integration suffices for all.

Many products can be integrated by first multiplying out; and some fractions by first dividing out. Others must be integrated by methods to be shown later. *They cannot be integrated by integrating the factors or numerator and denominator* separately.

A constant factor can be moved from one side of the sign $\int$ to the other side, inasmuch as it merely multiplies the result by so much in either position. This is not true for a variable factor.

Ex. I. Find $\int x^2\left(x+10\sqrt{x}+\frac{4}{x^2}+\frac{1}{x^5}\right)dx.$

Multiplying, and expressing every term as a *power* rather than a radical or fraction, we obtain the form

$$\int (x^3+10\,x^{\frac{5}{2}}+4+x^{-3})\,dx.$$

Applying formula (1) with $n=3$, $\frac{5}{2}$, 0, and -3 in turn, we get the integral:

$$\frac{1}{4}x^4+\frac{20}{7}x^{\frac{7}{2}}+4x-\frac{1}{2x^2}+C.$$

Remark. A term of the type $1/x$ in the integrand is usually not rewritten as a negative power (x^{-1}), being integrable by formula (2).

Ex. II. Find
$$\int\frac{(x^3+10)^2}{x^4}\,dx.$$

Expanding and dividing through by the denominator gives:

$$\int\left(x^2+\frac{20}{x}+100\,x^{-4}\right)dx.$$

Integrating gives:

$$\frac{1}{3}x^3+20\log x-\frac{100}{3x^3}+C.$$

§ 57. Elementary Applications. Of the various applications of integration mentioned in the *Introduction*, the following are listed here for present reference.

(*A*) *Motion of a point.* Integrating the acceleration a with respect to the time t gives the speed v. Integrating v with respect to t gives the distance s, measured along the path from some fixed point:

$$\int a\,dt=v, \qquad \int v\,dt=s.$$

Here v is considered negative in the direction in which s decreases. Also, in curvilinear motion, a denotes only the component of acceleration which acts along the path. This will be discussed in Chap. XI.

(*B*) *Equation of a plane curve.* Integrating the flexion with respect to x gives the slope; integrating the slope gives the ordinate y at any point.

(*C*) *Area under a curve.* The area between any continuous curve and the X-axis, from a fixed ordinate KL to a moving ordinate PQ, grows at a rate always equal to y; *i.e.*, $dA/dx=y=f(x)$. Hence

$$A=\int y\,dx. \tag{6}$$

Caution. This assumes that PQ moves to the right and that y is positive. If the curve lies below the X-axis, (6) gives the negative of the actual area, and the sign should finally be changed to plus. If the curve lies partly above and partly below the axis, we find each crossing, calculate separately the areas above and below, and add their actual amounts.

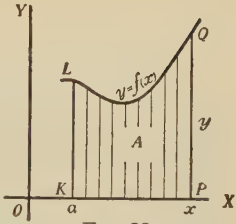


FIG. 33.

(*D*) *Volume of a solid* from a fixed plane to a parallel moving plane:

$$V = \int A_s \, dx. \tag{7}$$

Here A_s is the area of the moving cross section, and x its perpendicular distance from some fixed point.

(*E*) *Momentum generated* by a variable force F in the time t:

$$M = \int F \, dt. \tag{8}$$

(*F*) *Work done* by a variable force F in a distance x:

$$W = \int F \, dx. \tag{9}$$

(*G*) *Force of water pressure* against a vertical dam, of width w ft. at any depth x ft. below the surface:

$$F.P. = \int 62.5 \, xw dx. \tag{10}$$

Remarks. (I) The constant of integration obtained in using any of the formulas (6)–(10) is determined by the value of x or t at which the growing area, volume, etc., starts.

For instance, if the integration in (6) gives

$$A = F(x) + C, \tag{11}$$

and if $A = 0$ when $x = a$, then clearly $C = -F(a)$. The growing area is then

$$A = F(x) - F(a). \tag{12}$$

The integral is now definite and free from any ambiguity.

Substituting any other value b for x would give the area from $x=a$ to $x=b$, viz.

$$A=F(b)-F(a). \tag{13}$$

Likewise each other quantity in (7) (10), when calculated between definite boundaries, comes out a similar difference.

(II) To calculate any such quantity between definite boundaries, we need not work with a constant of integration, but simply find an integral function $F(x)$ and take the difference of its values at the boundaries, viz. $F(b)-F(a)$, as in (13).

Such a difference is often called a "definite integral from $x=a$ to $x=b$," and indicated by writing the boundary values of x after the integral sign, — with the starting value at the bottom and end value at the top.

Thus the area under the curve $y=3x^2$ from $x=5$ to $x=10$ is

$$A=\int_5^{10} 3x^2 dx.$$

And to work this out we find the integral x^3, get its values at 10 and 5 (viz. 1000 and 125), and subtract. Hence $A=875$.

EXERCISES

1. Find the following integrals (a, b, c, and k being constants):

(a) $\int\left(x^6+5x^{\frac{9}{2}}-\frac{x^2}{7}-4\right)dx,$ (b) $\int\left(\sqrt{x^3}-\frac{x}{9}\right)dx,$

(c) $\int\left(\frac{5}{t}+\frac{5}{t^2}-\frac{1}{4t^3}+\sqrt[3]{t}\right)dt,$ (d) $\int\left(\frac{4}{v}-\frac{4}{\sqrt{v}}\right)dv,$

(e) $\int\left(ax+\frac{1}{bx}-\frac{c}{\sqrt[3]{x}}\right)dx,$ (f) $\int\left(\frac{k}{x^2}+c-e^x\right)dx,$

(g) $\int\theta^2\left(4\theta^3-\frac{2}{\theta^3}+\frac{3}{2\theta^4}\right)d\theta,$ (h) $\int\frac{(5+\sqrt{u})^2}{3u^2}du,$

(i) $\int(\cos\phi-\sin\phi-1)d\phi,$ (j) $\int(10^t-t^{10})dt.$

(k) $\int(4e^x+2^x)dx,$ (l) $\int(4^x+x^4)dx.$

2. Verify the following integrations by differentiating the results:

(a) $\int 12\cos 300t\,dt=\frac{\sin 300t}{25}+C,$

(b) $\int 60(3^{-4x})dx=-\frac{15(3^{-4x})}{\log 3}+C.$

3. If $y=10$ when $x=0$ in each of the following cases, determine the constant of integration C:

(a) $y=\int(x^2+5)dx$, (b) $y=\int 4\,e^x dx$, (c) $y=\int \sin x\,dx$.

4. Find a formula for the height y of a curve, if the slope at any point is $3\,x^2$, and $y=8$ when $x=1$. Likewise if the flexion at any point is $12\,x$, and the slope and height at $x=0$ are respectively 2 and 5.

5. Find a formula for the distance traveled by an object, if the acceleration after t sec. is $30-.6\,t$, and if the speed at the start is 50.

6. Find the value of each of the definite integrals:

$$\int_1^4 \frac{3\,dx}{x}, \qquad \int_1^4 \frac{3\,dx}{x^2}, \qquad \int_1^4 \frac{3\,dx}{\sqrt{x}}, \qquad \int_1^4 \frac{\sqrt{x}}{3}\,dx.$$

7. Find the area under the parabola $y=x^2$ from $x=1$ to $x=3$. Check roughly by estimating the average height in that interval.

8. Find the area under each of the following curves:

(a) $y=e^x$, $x=0$ to 1; (b) $y^2=4\,x$, $x=1$ to 9;

(c) $y=\sin x$, 0 to $\frac{\pi}{2}$; (d) $xy=50$, 2 to 10.

9. Find the area bounded by the X-axis and that part of the curve $y=20\,x-x^2$ which lies above the axis.

10. Find the distance traveled by an object from $t=0$ to $t=3$ if the speed v varies thus:

(a) $v=70-32\,t$, (b) $v=4\,e^t$, (c) $v=10^{-t}$.

11. Find the momentum generated in the first 5 sec. by a force which varies thus:

(a) $F=12\,t-t^2$, (b) $F=10\cos t$, (c) $F=10-3\,e^{-t}$.

12. The force of attraction between two electrical charges x cm. apart varied thus: $F=60/x^2$. Find the work done in moving them apart, from $x=2$ to $x=8$.

13. Find the force of water pressure against a vertical dam, down to a depth of 10 ft., if the width w ft. varies thus:

(a) $w=100-x^2$, (b) $w=400-10\,x$, (c) $w=300-.1\,x^3$.

14. Every horizontal cross section of a solid, x ft. below the top, is a circle whose radius varies thus: $r=3+2\,x$. Find the volume, if the total height is 6 ft.

15. Find by integration the volume of a cone of height 6 and base radius 3. Check by elementary geometry.

16. The base of a solid is a quarter-circle of radius 3 in., and every vertical section parallel to one side is a right triangle whose height is twice its base. Find the volume.

17. The force driving a piston varies inversely as the distance x from a certain point; and $F=120$ when $x=5$. Find the work done from $x=2$ to $x=6$.

18. The slope of a curve varies as x, and is -8 at a point where $x=4$ and $y=10$. Find the area under the curve from $x=0$ to $x=6$.

§ **58. Fundamental Theorem.** Let $f(x)$ be any function which varies continuously with x, from $x=a$ to $x=b$. And let $f(x_1)$, $f(x_2)$, $\cdots$, $f(x_n)$, be values of the function taken at equal intervals Δx from a to b, — or at any values of x within such equal intervals. Denote by S the following sum:

$$S=f(x_1)\Delta x+f(x_2)\Delta x \cdots +f(x_n)\Delta x. \tag{14}$$

Also, denote by $F(x)$ an integral of $f(x)dx$, — that is, a function whose differential is $f(x)dx$.

Then we may state this extremely important *Theorem: If $\Delta x \to 0$, the sum S will approach a limit,— namely,*

$$F(b)-F(a).$$

In other words, the limit of S is precisely the difference which we have been calling the definite integral of $f(x)$ from $x=a$ to $x=b$.

This theorem can be proved rigorously by a purely algebraic argument of some length; but it is most easily seen geometrically. No matter what $f(x)$ means physically, we can plot its graph. (Fig. 34.) Then $f(x_1)$ is the height at x_1, etc.; and each product $f(x_1)\Delta x$, etc., is the area of a rectangle, — in a sense "inscribed" in a strip of base Δx.

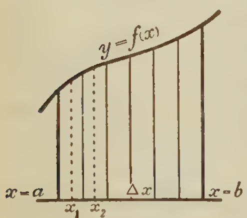


FIG. 34.

By hypothesis the curve is free from breaks. The sum of the rectangles approximates the area under the graph, and must approach the latter as a limit,

when $\Delta x \rightarrow 0$. By (13), p. 90, that area is $F(b)-F(a)$, — which, therefore, is the limit of the sum S, as the theorem states.

The limit of the sum of the rectangles, in fact, *defines* the area under the graph.*

Many geometrical and physical quantities can be expressed as the limit of a sum, of the type S. If the function $f(x)$ therein involved is continuous, and if we can find its integral function $F(x)$, the quantity in question can be calculated by merely forming the difference of two values of $F(x)$. The following example will illustrate.

Ex. I. A beam is unevenly loaded. The rate of loading (w lb. per ft.) varies thus with the distance (x ft.) from one end: $w=100+.3\,x^2$. Find the total weight on the beam between $x=2$ and $x=10$.

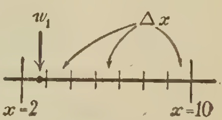


Fig. 35.

Let w_1 be the value of w at any chosen point in the first Δx portion; w_2 in the second; etc. Multiplying w_1 (the number of pounds per foot) by Δx (the fraction of a foot in the first portion) will give the approximate weight on that portion. The total weight W, from 2 to 10, is approximately the sum of $w_1\Delta x$, $w_2\Delta x$, etc. Exactly, W is the *limit* of this sum as $\Delta x \rightarrow 0$:

$$W = \underset{\Delta x \rightarrow 0}{L}\,[w_1\Delta x + w_2\Delta x \cdots + w_n\Delta x].$$

The sum here is of the type S, and w is a continuous function of x. Hence our theorem applies.

$$\therefore\ W=\int_2^{10} w\,dx=\int_2^{10}(100+.3\,x^2)\,dx.$$

*Cf. *Intro.*, § 38.

Integrating, substituting, and subtracting:

$$W = 100\,x + .1\,x^3\Big]_2^{10} = 1100 - 200.8 = 899.2.$$

The bracket after the integral function denotes the difference of two values of that function.

A somewhat more abbreviated method of setting up integrals will be discussed presently. (§ 68.)

§ **59. Another Definition of an Integral.** As shown in § 58, when $f(x)$ is continuous between $x=a$ and $x=b$, the limit of the sum S in (14) is

$$\underset{\Delta x \to 0}{\mathrm{L}}\,[f(x_1)\Delta x + \cdots + f(x_n)\Delta x] = F(b) - F(a). \qquad (15)$$

If the upper boundary of summation, b, were changed to any other value x, the right member of (15) would become $F(x) - F(a)$.

That is, *the limit of the sum* S, using a variable upper boundary of summation x, *is in very fact the integral function,* $F(x)+C$, — with the constant determined as $-F(a)$.

In other words, the limit of the sum depends upon x in such a way, or is such a function of x, that its derivative would be $f(x)$.

Hence, although we have thus far defined an integral as the inverse of a derivative or differential, we may equally well define it as the limit of a sum, — viz. the sum in (15), but with the upper boundary at x rather than b.

And, since the constant is determined and there is no ambiguity in the limit of the sum, we shall call this a "definite integral" and write it

$$\int_a^x f(x)\,dx. \qquad (16)$$

In the case of a continuous function, then, it is immaterial whether we regard the words "definite integral" and the symbol in (16) as standing for the limit of a sum or for the difference of two values of an integral function obtained by reversing the differentiation process. But in cases involving

discontinuities it will be more satisfactory to adopt the limit of the sum as the real definition of a definite integral.

Consequently, when the sum S fails to approach any limit, we shall say that the definite integral does not exist. On the other hand, *whenever S does approach a limit, we shall say that $f(x)dx$ has an integral* (viz. that limit), even if we can find no standard function to express the integral exactly. Thus $f(x)$ may be integrable in this fundamental sense without being integrable in the elementary sense.

Remark. The boundaries of summation a and x, or a and b, are also called the "limits of integration." (This use of the word "limit" is, of course, different from its other use to signify a limiting value approached by a variable quantity.)

EXERCISES

1. Set up formula (7), p. 89, by reasoning as in Ex. I, § 58. (Consider n slices and express V as the limit of a sum.)

2. Likewise set up formulas (8) and (9), p. 89.

3. In each of the following cases express the quantity in question as the limit of a sum, and then in integral form:

(*a*) The total weight of a pole 10 ft. long, if the weight per ft. varies with the distance (x ft.) from one end;

(*b*) The load on a rectangular floor 15 ft. wide and 40 ft. long if the load per sq. ft. varies with the distance (x ft.) from one end.

(*c*) The normal increase in bodily weight between the ages of 10 and 15, if the normal rate of increase per yr. varies continually.

(*d*) The normal consumption of sugar during April and May, if the rate of consumption per month varies continually with the time (t mo.) since Jan. 1.

(*e*) The total energy consumed by a factory, from $t=2$ to $t=4$, if the power varied with the time (t hr.) after starting. [Power is the rate of using energy.]

(*f*) The volume of water flowing over a dam, from $t=3$ to $t=7$, if the rate of flow per hr. varies with the time (t hr.) since starting.

(*g*) The volume of water evaporated from a lake in April if the rate of evaporation per day varies with the time (t days) since the beginning of the month.

4. A quantity Q is the following limit of a sum:

$$Q = \underset{\Delta x \to 0}{\mathrm{L}} [x_1^3 \Delta x + x_2^3 \Delta x \cdots + x_n^3 \Delta x],$$

where x_1, x_2, etc., are values taken anywhere in n successive equal intervals Δx between $x = a$ and $x = b$. Express Q also in integral notation, and find its value.

5. As in Ex. 4, find each of the following limits, for the interval from $x = 1$ to $x = 5$:

(a) $\underset{\Delta x \to 0}{\mathrm{L}} [x_1^2 \Delta x + x_2^2 \Delta x \cdots + x_n^2 \Delta x]$,

(b) $\underset{\Delta x \to 0}{\mathrm{L}} \left[\frac{1}{x_1} \Delta x + \frac{1}{x_2} \Delta x \cdots + \frac{1}{x_n} \Delta x\right]$,

(c) $\underset{\Delta x \to 0}{\mathrm{L}} [e^{x_1} \Delta x + e^{x_2} \Delta x \cdots + e^{x_n} \Delta x]$.

6. Calculate the sum in Ex. 5 (a) when $\Delta x = 2$, $(n = 2)$, taking each x in the middle of its interval. How much does the sum differ from the limit found in 5 (a)? Repeat, using $\Delta x = 1$, $(n = 4)$.

Ex. 7–11 are for further practice on §§ 55–57.

7. Find the values of the following definite integrals:

(a) $\displaystyle\int_1^8 \frac{dx}{7x}$, $\displaystyle\int_1^8 \frac{dx}{7x^3}$, $\displaystyle\int_1^8 \frac{dx}{7\sqrt[3]{x}}$, $\displaystyle\int_1^8 \frac{x}{7}\,dx$;

(b) $\displaystyle\int_2^4 \frac{(2\sqrt{x}+3)^2\,dx}{5x}$; (c) $\displaystyle\int_{10}^{20} \frac{3x^4+5x+4}{2x^2}\,dx$;

(d) $\displaystyle\int_0^3 (2^x + 2^{-x})\,dx$; (e) $\displaystyle\int_2^3 x^{19}\,dx$.

8. In a certain solid the cross section area varies as the cube of its distance below the top; and $A_s = 20$ when $x = 2$. Find the volume, from $x = 1$ to $x = 4$.

9. The flexion of a curve at any point (x, y) is $2 \sin x$. At $x = 0$ both y and the slope are zero. Find the area under the curve, from $x = 0$ to $x = \frac{\pi}{2}$.

10. The magnetic potential between two co-axial cylindrical shells is $V = \int_a^b E\,dr$, where $E = 4\pi k\,a/r$. Work out the integral.

11. Verify integration formulas (1), (2), (3), p. 87, by differentiating the right-hand members.

§ 60. Some Properties of Definite Integrals. As we have seen, any continuous function $f(x)$ has an integral $F(x)$; and

$$\int_a^b f(x)\,dx = F(b) - F(a). \qquad (17)$$

Let us now note several further facts concerning definite integrals, which will be useful later.

(1) *Interchanging the limits a and b changes the sign of the result.* This is clear from (17). It also follows from the definition as the limit of a sum. For if Δx is positive going from a to b, it is negative going from b to a.

(2) *A definite integral is a function of its limits and not of the variable used in integrating.* In fact, the latter variable disappears on substituting the limits.

Thus $$\int_0^{\frac{\pi}{2}} \cos t\, dt \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos z\, dz$$

are equal regardless of any relation which might exist between t and z.

(3) *Splitting the interval:*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx. \tag{18}$$

This follows either from (17) or from the definition of an integral as the limit of a sum.

(4) *Negative Integrands.* If $f(x)$ is negative from $x=a$ to $x=b$ (except at any points where it may be zero) and if $b > a$, then

$$\int_a^b f(x)\, dx \quad \text{is negative.}$$

Further, if $f(x)$ is positive in some parts of the interval and negative in the others, the $+$ and $-$ terms in the summation cancel to some extent and tend to make the integral small.

(5) *Inequalities.* If in the two integrals

$$\int_a^b f_1(x)\, dx \quad \text{and} \quad \int_a^b f_2(x)\, dx,$$

$f_1(x)$ is everywhere positive and greater than the numerical value of $f_2(x)$, — except at any points where they are equal, — then the first integral is greater than the second.

For the terms of the first sum are greater than those of the second, even if the latter are all of like sign and do not cancel one another. Graphically speaking, the area under the f_1 curve is greater than that between the f_2 curve and the X-axis. (Fig. 36.)

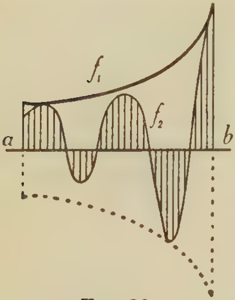


FIG. 36.

§ 61. Generalized Meaning of Formulas. The basic integration formulas cover a far wider variety of expressions than is at first apparent. *E.g.*, in the formula

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad (n \neq -1)$$

u need not be a simple variable, such as x, t, or θ; it may be any *quantity* depending upon x, or θ, etc. But then du denotes the differential of that same quantity; and if that differential is not present in a given integrand, the formula does not apply.

To see whether a given expression is actually covered by some basic formula, look first for some part of the expression which might be the differential of the remaining part or of some quantity involved in the remaining part in a simple way.

Ex. I. $$\int \sin^5 x \cos x \, dx.$$

We know that $\cos x\, dx$ is the differential of $\sin x$. Regard $\sin x$ as u. Then $\sin^5 x$ is simply u^5 and $\cos x\, dx$ is du. Thus the entire integrand is of the form $u^5 du$; and integrating gives $\frac{1}{6} u^6 + C$. Hence, remembering what u stands for:

$$\int \sin^5 x \cos x \, dx = \tfrac{1}{6} \sin^6 x + C. \tag{19}$$

Check. Differentiating $(\sin x)^6$ would give $6 (\sin x)^5 \cos x\, dx$.

Ex. II. $$Q = \int \frac{x^2 dx}{8 - 5x^3}.$$

Here $x^2\, dx$ is the differential of the entire denominator aside from a numerical factor -15. We supply this factor, and

compensate by placing its reciprocal $-\frac{1}{15}$ before the integral sign:

$$Q=-\tfrac{1}{15}\int\frac{-15\,x^2\,dx}{8-5\,x^3}. \tag{20}$$

The integrand is now of the form du/u. Hence

$$Q=-\tfrac{1}{15}\log(8-5\,x^3)+C. \qquad \text{(Check?)}$$

Only a *constant* factor can be multiplied in and divided out, on opposite sides of the integral sign, as was done here.

EXERCISES

1. Observe whether each of these integrals is already of a type form and carry out each integration:

$$\int(x+5)^{10}\,dx,\quad \int 4\,x^3(x^4-1)^6\,ax,\quad \int\sin^8\theta\cos\theta\,d\theta,\quad \int\tan^4\theta\sec^2\theta\,d\theta.$$

2. Integrate each of the following quantities:

(*a*) $(3\,x+1)^5\,dx$, $(kx-3)^{\frac{7}{3}}\,dx$, $\sqrt[5]{9-x}\,dx$, $\sqrt{2\,ax}\,dx$, $(x^2+1)^{10}\,x\,dx$;

(*b*) $e^{-4x}dx$, $\sin 20\,t\,dt$, $2\cos 5\,\theta\,d\theta$, $10^{.06t}dt$, $e^{\sin x}\cos x\,dx$;

(*c*) $(6\sin 4\,\theta-5\cos 4\,\theta)d\theta$, $(e^{6v}+e^{-6v})dy$, $3^{2x}(1+3^{-4x})dx$;

(*d*) $\dfrac{dx}{6\,x-5}$, $\dfrac{x^3\,dx}{a^4-x^4}$, $\dfrac{x^3\,dx}{(a^4-x^4)^2}$, $\dfrac{dx}{\sqrt{10+3\,x}}$, $\dfrac{x\,dx}{x^2+16}$,

(*e*) $\dfrac{e^{3x}dx}{e^{3x}+1}$, $\dfrac{e^{3x}\,dx}{(e^{3x}+1)^4}$, $\dfrac{e^{3x}\,dx}{\sqrt[4]{e^{3x}+1}}$, $\dfrac{\sin\theta\,d\theta}{1-\cos\theta}$, $\dfrac{\sin\theta\,d\theta}{\sqrt{1-\cos\theta}}$,

(*f*) $(\log x)^4\dfrac{dx}{x}$, $\sqrt{\log x}\,\dfrac{dx}{x}$, $\dfrac{\cos(\log x)\,dx}{x}$, $\dfrac{dx}{x\log x}$, $\dfrac{e^x\,dx}{\sec e^x}$.

3. Evaluate (*i.e.*, find the value of) each of the definite integrals:

(*a*) $\displaystyle\int_0^{\frac{\pi}{4}}\tfrac{1}{2}(1-\cos 2\,\theta)d\theta$, (*b*) $\displaystyle\int_0^1(e^x+e^{-x})^3dx$,

(*c*) $\displaystyle\int_0^a x\sqrt{a^2-x^2}\,dx$, (*d*) $\displaystyle\int_{-2}^4\frac{dy}{8-y}$,

(*e*) $\displaystyle\int_0^{\frac{\pi}{2}}\cos^4 t\sin t\,dt$, (*f*) $\displaystyle\int_{-1}^1\frac{5^{-x}}{2^x}\,dx$,

(*g*) $\displaystyle\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\text{ctn}^3\,2\,\theta\csc^2 2\,\theta d\theta$, (*h*) $\displaystyle\int_a^b\frac{\cos t\,dt}{\sin^{\frac{3}{2}}t}$.

4. The force moving an object varied thus: $F=2\cos 4\,\pi t$. Find the momentum generated from $t=0$ to $t=\frac{1}{8}$.

5. Evaluate the following integral which arises in the study of diffraction gratings [π, λ, and θ being constants]:

$$A=\int_{-\frac{s}{2}}^{\frac{s}{2}}\cos\left(\frac{2\,\pi\theta}{\lambda}\right)x\,dx.$$

6. The rate at which heat enters the ground varies thus during a daily cycle: $dH/dt=k\,(\sin\omega t+\cos\omega t)$, where k and ω are constants. Find a formula for the amount H entering during the first half-cycle, $t=0$ to $t=\pi/\omega$. Also verify that $t=\pi/\omega$ is correct for a half-cycle. (How large is the phase angle ωt at the end of a complete cycle?)

7. The intensity of an electric current (or rate of flow) varied thus: $i=10\sin 100\,\pi t$. Find the quantity of electricity passed in the first half-cycle, $t=0$ to $t=.01$. How much in the second half-cycle, $t=.01$ to $.02$? Interpret the negative sign.

8. A cylindrical tank of diameter 8 feet, with its axis horizontal, is half full of water. Find the force of water pressure against one end.

9. (*a*) Find the area inclosed by the X-axis and the parabola $y=x^2-10\,x$. (*b*) Likewise the entire area inclosed by the X-axis and the curve $y=x^3-6\,x^2+5\,x$. Plot and check roughly.

10. (*a*) Before integrating show that $\int_0^1 x^3dx<\int_0^1 x^2dx$. Verify.

(*b*) Verify also the following equalities:

$$\int_4^1 x^2dx=-\int_1^4 x^2dx;\qquad \int_1^4 x^2dx+\int_4^6 x^2dx=\int_1^6 x^2dx.$$

11. Show by a very brief argument that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos x}{x}dx<\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{x}\,dx;$$

and hence that the first integral must be less than log 2.

12. Show similarly that

$$\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\,dx<\int_0^{\frac{\pi}{2}}\sin^{2n}x\,dx<\int_0^{\frac{\pi}{2}}\sin^{2n-1}x\,dx.$$

[From this fact an important formula for π is deducible.]

13. Prove briefly the following inequalities:

(*a*) $\int_0^1 e^{-x}x^{.4}dx>\int_0^1\frac{x^{.4}}{e}\,dx$, and hence $>\frac{1}{1.4\,e}$;

(*b*) $\int_0^{20}e^{-x}x^{.4}dx>\int_0^{19}e^{-x}x^{.4}dx$;

(*c*) $\int_0^{\pi}\sin x\,dx>\int_0^{\phi}\sin x\,dx$, if ϕ lies between π and $2\,\pi$.

§ 62. Further Trigonometric Types. We have seen how to integrate the sine and cosine. The other basic functions are covered by the following formulas:

$$\int \tan u\,du = -\log \cos u \;+C, \tag{21}$$

$$\int \operatorname{ctn} u\,du = \log \sin u \;+C, \tag{22}$$

$$\int \sec u\,du = \log(\sec u+\tan u) \;+C, \tag{23}$$

$$\int \csc u\,du = -\log(\csc u+\operatorname{ctn} u) \;+C. \tag{24}$$

Integrals of two important products are:

$$\int \sec u \tan u\,du = \sec u \;+C, \tag{25}$$

$$\int \csc u \operatorname{ctn} u\,du = -\csc u \;+C. \tag{26}$$

These two, and (27), (28) below, follow directly from differentiation formulas.

Derivation of Formulas (21)–(24)

To derive (21) write $\tan u\,du$ as $\sin u\,du/\cos u$, which is of the type $-dv/v$, where v is $\cos u$. Similarly for (22).

To derive (23), multiply and divide by $(\sec u+\tan u)$, getting

$$\int \sec u\,du = \int \frac{(\sec^2 u+\sec u \tan u)du}{\sec u+\tan u}.$$

The numerator is now the differential of the denominator, with the terms in reverse order. Thus the type is again dv/v. Similarly for (24).

Squares of the Functions. The square of each basic trigonometric function also is easily integrated.

$$\int \sec^2 u\,du = \tan u + C, \tag{27}$$

$$\int \csc^2 u\,du = -\operatorname{ctn} u + C. \tag{28}$$

To integrate $\tan^2 u$ express it as $\sec^2 u-1$. Similarly, express $\operatorname{ctn}^2 u$ as $\csc^2 u-1$.

To integrate $\sin^2 u$, express it in terms of the double angle $2u$, viz.

$$\mathbf{\sin^2 u = \tfrac{1}{2}(1 - \cos 2u).} \tag{29}$$

Integrating gives $\frac{1}{2}u - \frac{1}{4}\sin 2u + C$. Similarly for $\cos^2 u$, write

$$\mathbf{\cos^2 u = \tfrac{1}{2}(1 + \cos 2u),} \tag{30}$$

and integrate.

Memorize carefully formulas (21)–(30). This is easily done by concentrating upon those in black-face type, and comparing the others. Knowing the derivations also helps.

We need to have at our finger tips the integral of each trigonometric function and of each product above; also the procedure for integrating the square of each function.

EXERCISES

1. Integrate each of the following quantities:

(*a*) $\tan 4x\,dx$, $\quad \sec 6\theta\,d\theta$, $\quad \operatorname{ctn} kt\,dt$, $\quad \csc ay\,dy$;

(*b*) $\sin^2 3x\,dx$, $\quad \cos^2 \frac{\phi}{3}\,d\phi$, $\quad \sec^2 \frac{2u}{3}\,du$, $\quad \operatorname{ctn}^2 7t\,dt$;

(*c*) $\dfrac{dx}{\cos 5x}$, $\quad \dfrac{dy}{\sin^2 2y}$, $\quad \dfrac{d\theta}{\operatorname{ctn}^2 4\theta}$, $\quad \dfrac{d\phi}{\csc \frac{\phi}{2}}$;

(*d*) $\tan ax \sec ax\,dx$, $\quad \operatorname{ctn} 2x \csc 2x\,dx$, $\quad \sin 8x \cos 8x\,dx$,

(*e*) $\tan^5 x \sec^2 x\,dx$, $\quad \operatorname{ctn} 2x \csc^2 2x\,dx$, $\quad (1+\cos 2x)^2 dx$,

(*f*) $\sec^2 (\log x) dx/x$, $\quad \operatorname{ctn} (3^x) 3^x\,dx$, $\quad (1-\sin 4x)^2\,dx$.

2. Evaluate each of the definite integrals:

(*a*) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 \theta\,d\theta$, $\quad$ (*b*) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{ctn}^2 t\,dt$, $\quad$ (*c*) $\displaystyle\int_0^{\frac{\pi}{8}} \sec 2x\,dx$,

(*d*) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta\,d\theta}{\sin^2\theta}$, $\quad$ (*e*) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x\,dx}{\tan x}$, $\quad$ (*f*) $\displaystyle\int_0^{\frac{\pi}{4}} e^{\tan t} \sec^2 t\,dt$.

3. Find the integral of vers $u\,du$. [Hint: What is the meaning of vers u?]

4. Integrate $\tan u$ by first multiplying and dividing it by $\sec u$, and regarding $\sec u$ as v. Reconcile your result with (21).

5. By using the double-angle formulas twice transform $\sin^4 x$ into $\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$. Then integrate.

6. Find the area under the tangent curve, $y=\tan x$, $x=0$ to $x=\frac{\pi}{4}$.

7. Find the area under one arch of the common cycloid. [Hint: Express $y\,dx$ in terms of ϕ and $d\phi$. How far does ϕ run to complete one arch? (§ 19.)]

8. The heating effect of an electric current is proportional to $\int i^2\,dt$. Evaluate this integral for a half-cycle if $i=20\sin 120\,\pi t$.

9. Every horizontal section of a globe, x in. above the lowest point, is a circle whose radius is $4\sin(\pi x/20)$ in. Find its volume.

10. In determining the number of free positive electrons in the nucleus of an atom by Rutherford's method, it is necessary to find a quantity m given by

$$m=-\frac{\pi}{4}ntb^2\int_{\phi_1}^{\phi_2}\operatorname{ctn}\frac{\phi}{2}\csc^2\frac{\phi}{2}\,d\phi.$$

Evaluate this integral, and so get a formula for m.

§ 63. Further Algebraic Types. From the results of several earlier differentiations (Ex. 3, p. 84), we obtain immediately the following important formulas:

$$\int\frac{du}{u^2+a^2}=\frac{1}{a}\tan^{-1}\frac{u}{a}+C, \tag{31}$$

$$\int\frac{du}{u^2-a^2}=\frac{1}{2a}\log\frac{u-a}{u+a}+C, \tag{32}$$

$$\int\frac{du}{\sqrt{u^2\pm a^2}}=\log(u+\sqrt{u^2\pm a^2})+C. \tag{33}$$

Notes

In (33) the sign $+$ goes with the $+$, and the $-$ with the $-$.

In the right member of (32) we may write $(a-u)$ in place of $(u-a)$. Multiplying there by -1 merely adds $\log(-1)$, an imaginary constant. Which form would be better in a concrete problem depends upon whether a or u is the larger. But if the denominator on the left side were given as a^2-u^2, this would change the sign of the entire right member. (Cf. Ex. II below.)

In memorizing these formulas it will be helpful to contrast the forms carefully, *e.g.*, the first two which involve no radical, and to note the effect of introducing a radical as in (33).

These formulas, like all the others, have a generalized meaning. In particular, they can be applied to various trinomial quadratic expressions by completing the square. (Cf. Ex. III below.)

Ex. I. $$Q=\int\frac{\sec^2 x\,dx}{3\tan^2 x+49}.$$

This suggests $\dfrac{du}{u^2+a^2}$ with $u=\sqrt{3}\tan x$, $a=7$. But the du then needed would be $\sqrt{3}\sec^2 x\,dx$.

We supply the factor $\sqrt{3}$ above, and compensate outside.

$$Q=\frac{1}{\sqrt{3}}\int\frac{\sqrt{3}\sec^2 x\,dx}{3\tan^2 x+49}=\frac{1}{7\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{7}\right)+C.$$

Ex. II. $$Q=\int\frac{8\,dx}{25-9x^2}.$$

This resembles (32) with $u=3x$ and $a=5$. But the signs are reversed; and instead of $8\,dx$ in the numerator we need $3\,dx$ as du.

We make the form standard by rewriting it thus:

$$Q=-\frac{8}{3}\int\frac{3\,dx}{9x^2-25}.$$

This now comes under (32), and integrating gives:

$$Q=-\frac{8}{3}\cdot\frac{1}{10}\log\frac{3x-5}{3x+5}+C, \tag{34}$$

Or, what is more convenient, if $3x<5$ in a given problem,

$$Q=-\frac{4}{15}\log\frac{5-3x}{5+3x}+C'. \tag{35}$$

The sign $-$ before either (34) or (35) can be changed to $+$ if we invert the fraction whose logarithm is involved. Why?

Ex. III. $$\int \frac{dx}{\sqrt{x^2+7x+6}}.$$

The completed square for x^2+7x is $x^2+7x+\frac{49}{4}$. The given quantity under the radical is less than this by $\frac{25}{4}$, and is therefore $(x+\frac{7}{2})^2-\frac{25}{4}$, or u^2-a^2. Also dx is du. Thus we have the type (33) and the integral is $\log(u+\sqrt{u^2-a^2})$, or

$$\log\left(x+\tfrac{7}{2}+\sqrt{x^2+7x+6}\right) \quad +C.$$

We could also write this: $\log(2x+7+2\sqrt{x^2+7x+6})+C'$. For multiplying throughout the parenthesis by 2 merely adds $\log 2$, a constant.

EXERCISES

1. In each of the following forms consider the numerator as du, aside from some numerical factor; and try then to choose u so that the expression will be a type form. Integrate each.

(a) $\dfrac{dx}{4x^2-9}$, $\dfrac{x^4dx}{x^{10}+9}$, $\dfrac{dx}{\sqrt{9x^2-16}}$, $\dfrac{dx}{8-5x^2}$, $\dfrac{x^3dx}{\sqrt{16+x^8}}$;

(b) $\dfrac{t^3dt}{t^4+25}$, $\dfrac{tdt}{t^4+25}$, $\dfrac{tdt}{\sqrt{t^4+25}}$, $\dfrac{t^3dt}{\sqrt{9t^4-25}}$, $\dfrac{tdt}{9t^4-25}$;

(c) $\dfrac{e^xdx}{\sqrt{e^{2x}-7}}$, $\dfrac{e^{2x}dx}{\sqrt{e^{2x}-7}}$, $\dfrac{e^xdx}{e^{2x}-7}$, $\dfrac{e^xdx}{3e^{2x}+7}$. $\dfrac{e^{2x}dx}{3e^{2x}+7}$.

2. Evaluate the definite integrals:

(a) $\displaystyle\int_0^2 \frac{dy}{16-y^2}$, (b) $\displaystyle\int_1^e \frac{dx}{x[4-(\log x)^2]}$, (c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos\theta\, d\theta}{a^2\sin^2\theta+b^2}$,

(d) $\displaystyle\int_1^5 \frac{dx}{\sqrt{2x^2-1}}$, (e) $\displaystyle\int_0^5 \frac{e^u du}{e^{2u}+25}$, (f) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 t\, dt}{\sqrt{\tan^2 t+3}}$.

3. Write each of these expressions as the sum or difference of two fractions and then integrate:

(a) $\dfrac{(5x+3)dx}{3x^2-25}$, (b) $\dfrac{(6x^3-2x)dx}{2x^4+9}$, (c) $\dfrac{(7-x)dx}{\sqrt{4x^2-25}}$.

4. Integrate each of the following, after completing the square:

$\dfrac{dx}{x^2-4x+13}$, $\dfrac{dx}{x^2+4x}$, $\dfrac{dx}{\sqrt{x^2+5x+7}}$, $\dfrac{dx}{15-2x-x^2}$.

5. The age (t mo.) at which an average infant reaches a weight of W kg. is probably given by an integral of the form

$$t = k\int_a^W \frac{dx}{2\,ax - x^2}.$$

Show that this leads to the formula $t = -\frac{k}{2\,a}\log\frac{2\,a - W}{W}$, or

$$W = \frac{2\,a}{1 + e^{-\frac{2\,at}{k}}}.$$

6. When a wire weighing .2 lb./ft. is hung from two poles with a horizontal tension of 20 lb., the horizontal distance (x ft.) of any point P from the center is related thus to the slope L at P:

$$x = 100\int_0^L \frac{du}{\sqrt{1+u^2}}.$$

Deduce from this the formula $L = \frac{1}{2}(e^{.01x} - e^{-.01x})$. Then integrate the slope to find the equation of the curve of sag.

64. Type Forms Concluded. Three more formulas will complete our memory list of basic integrals:

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C, \tag{36}$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C, \tag{37}$$

$$\int \frac{du}{\sqrt{2\,au - u^2}} = \operatorname{vers}^{-1}\frac{u}{a} + C. \tag{38}$$

Contrast (36) with (33), p. 103. Also observe that the right-hand members in (31)–(33) and in (36)–(38) do not involve a coefficient $\frac{1}{a}$ or $\frac{1}{2\,a}$ when the original denominator is entirely within the radical.

It is well to collect for reference all the basic integral forms. Then study the list carefully and fix it so thoroughly in mind that you can tell on looking at any ordinary integrand what kind of function the integration will lead to: whether a logarithm, arcsine, or what.

§ 65. Splitting a Fraction. Some fractions are integrated by separating them into two other fractions, each a type form. If the denominator is a trinomial quadratic or a power thereof, we proceed as in the following example.

Ex. I. $$F=\int\frac{(3x+4)\,dx}{\sqrt{x^2+7x+6}}.$$

The differential of x^2+7x+6 is $(2x+7)dx$. We take this as one numerator, but multiplied by $\frac{3}{2}$ to preserve the original $3x$. Then we write

$$F=\tfrac{3}{2}\int\frac{(2x+7)\,dx}{\sqrt{x^2+7x+6}}+A\int\frac{dx}{\sqrt{x^2+7x+6}}, \tag{39}$$

where A is an unknown constant, to be determined.

In other words, as our first fraction we take *what we should like to have*, and to it we add *whatever we must* in order to make the sum of the two fractions equal to the given fraction.

Comparing numerators: $\frac{3}{2}(2x+7)+A=3x+4$. This gives $A=-\frac{13}{2}$.

The first integrand in (39) is of the form $dv/\sqrt{v}$, or $v^{-\frac{1}{2}}dv$. The second, upon completing the square as in Ex. III, p. 105, takes the form $du/\sqrt{u^2-a^2}$. Hence finally

$$F=\tfrac{3}{2}[2v^{\frac{1}{2}}]-\tfrac{13}{2}\log\left(u+\sqrt{u^2-a^2}\right)\quad+C;$$

i.e., $$F=3\sqrt{x^2+7x+6}-\tfrac{13}{2}\log\left(x+\tfrac{7}{2}+\sqrt{x^2+7x+6}\right)+C.$$

EXERCISES

1. Verify formulas (36)–(38) by differentiating the right members.

2. Make a list of the integrals of the following:

$$\frac{du}{u},\quad \frac{du}{\sqrt{u}},\quad \frac{du}{u^2+a^2},\quad \frac{du}{u^2-a^2},\quad \frac{du}{\sqrt{u^2-a^2}},\quad \frac{du}{\sqrt{a^2-u^2}},\quad \frac{du}{\sqrt{u^2+a^2}},\quad \frac{du}{u\sqrt{u^2-a^2}}.$$

Also name the distinguishing feature between integrands which are most nearly alike in this list, and mention the kind of function to which each leads when integrated.

3. Integrate each of the following forms:

(a) $\dfrac{6\,dx}{\sqrt{4-x^2}}$, $\dfrac{7\,dx}{5\sqrt{3-2\,x^2}}$, $\dfrac{3\,dx}{x\sqrt{x^2-4}}$, $\dfrac{9\,dx}{5\,x\sqrt{4\,x^2-1}}$;

(b) $\dfrac{5\,dx}{\sqrt{6\,x-x^2}}$, $\dfrac{dx}{2\sqrt{x-x^2}}$, $\dfrac{dx}{4\sqrt{3\,x-5\,x^2}}$ $\left[=\dfrac{dx}{4\sqrt{5}\sqrt{\frac{3}{5}\,x-x^2}}\right]$;

(c) $\dfrac{e^x\,dx}{\sqrt{9\,e^x-e^{2x}}}$, $\dfrac{y\,dy}{\sqrt{9-y^4}}$, $\dfrac{\sin\theta\,d\theta}{\sqrt{9-4\cos^2\theta}}$, $\dfrac{\cos\phi\,d\phi}{\sin\phi\sqrt{\sin^2\phi-.25}}$.

4. Evaluate the definite integrals:

(a) $\displaystyle\int_0^1 \frac{x^2dx}{\sqrt{1-x^6}}$; (b) $\displaystyle\int_1^2 \frac{dr}{5\,r\sqrt{2\,r^2-1}}$; (c) $\displaystyle\int_0^1 \frac{dy}{\sqrt{4\,y-3\,y^2}}$.

5. Integrate after completing the squares:

(a) $\dfrac{dx}{\sqrt{6\,x-x^2}}$; (b) $\dfrac{dy}{\sqrt{13+12\,y-y^2}}$; (c) $\dfrac{dt}{\sqrt{5+t-4\,t^2}}$.

6. Transform into type forms and then integrate:

(a) $\dfrac{\cos\phi\,d\phi}{5-\cos^2\phi}$; (b) $\dfrac{\sin\phi\,d\phi}{8+\sin^2\phi}$; (c) $\dfrac{\sec^2\phi\,d\phi}{\sqrt{\sec^2\phi+1}}$.

7. Split up and integrate:

(a) $\dfrac{(2\,x+13)dx}{4\,x^2-12\,x+25}$; (b) $\dfrac{(3\,x+5)dx}{\sqrt{5-4\,x-x^2}}$; (c) $\dfrac{(5\,x-4)dx}{\sqrt{6\,x-x^2}}$;

(d) $\dfrac{(2-7\,x)dx}{x\sqrt{x^2-25}}$; (e) $\dfrac{(ax+b)dx}{\sqrt{k^2-x^2}}$; (f) $\dfrac{x\,dx}{\sqrt{x^2+ax+b}}$.

8. Without working out, state what *kind* of function each of these integrals will be:

$$\int\frac{x^4\,dx}{(x^5+1)^2},\quad \int\frac{x\,dx}{x^2+20},\quad \int\frac{x^2\,dx}{\sqrt{x^3-1}},\quad \int\frac{x^2\,dx}{\sqrt{x^6-1}},$$

$$\int\frac{e^t\,dt}{e^{2t}+5},\quad \int\frac{y\,dy}{\sqrt{1-y^4}},\quad \int\frac{x\,dx}{\sqrt{1-x^2}},\quad \int\frac{\sin\phi\,d\phi}{1-4\cos^2\phi}.$$

§ 66. Consistency Tests. An integration, if performed in two different ways, may give results having different forms. But these should be *consistent*, *i.e.*, differ in value only by a constant.

Sometimes one result can be reduced to another by a simple transformation. The following devices often help.

(1) Algebraic forms: Multiply or divide out, or factor.

(2) Logarithms: Use the basic laws concerning the logarithm of a product, quotient, power, or root.

(3) Trigonometric forms: Transform all unlike terms into some one function, say $\sin x$, or $\tan x$, etc.

(4) Inverse trigonometric forms: Draw a right triangle and read off relations; or use the general relations connected with reciprocals or with complementary angles.

Thus $\arccos x$ is the same as $\operatorname{arcsec}(1/x)$; also it is the same as $\pi/2 - \arcsin x$. Similarly

$$\operatorname{ctn}^{-1} x = \tan^{-1}(1/x), \quad \text{also} \quad \pi/2 - \tan^{-1} x;$$
$$\csc^{-1} x = \sin^{-1}(1/x), \quad \text{also} \quad \pi/2 - \sec^{-1} x.$$

However, *when x is negative* and the functions are restricted to their principal values (§ 37), there is one change:

$$\operatorname{ctn}^{-1} x = \tan^{-1}(1/x) + \pi, \quad \text{also} \quad \pi/2 - \tan^{-1} x. \tag{40}$$

Double-angle formulas, etc., can be written also as relations among the inverse functions; but the angles involved may fall outside the respective principal values. Thus the relation may not apply to integrals which are expressed in terms of such principal values.

Sometimes one form of an integral is not readily reduced to another. To test the consistency in such a case, *subtract one form from the other, simplify, and see whether the difference reduces to a constant.*

If reductions are baffling, substitute several values for the variable and see whether the difference is the same for all. (This is not a certain test but is usually adequate.) Or, as a last resort, differentiate the difference and see whether its derivative is zero.

The following examples illustrate, in fairly complicated cases, the two principal methods: reduction and subtraction.

Ex. I. Reduce one of the following forms to the other:

$$(A)\quad (x^2+1)^{\frac{3}{2}}-3\sqrt{x^2+1}\quad +2\sin^{-1}\frac{x}{\sqrt{x^2+1}};$$

$$(B)\quad (x^2-2)\sqrt{x^2+1}\quad +\tan^{-1}\frac{2\,x}{1-x^2}.$$

The first two terms of (A) have a common factor $\sqrt{x^2+1}$. With this removed, there would remain $(x^2+1)-3$; that is, x^2-2, as in (B). These parts of (A) and (B) match.

Next we let the angle in (A) be called ϕ:

$$\sin^{-1}\frac{x}{\sqrt{x^2+1}}=\phi,\quad\text{or}\quad \frac{x}{\sqrt{x^2+1}}=\sin\phi; \tag{41}$$

and inquire whether $2\,\phi$ is the same as the angle in (B), whose tangent is $2\,x/(1-x^2)$.

By drawing a triangle (Fig. 37) to fit (41), we see that $\tan\phi=x$, simply. But by a double-angle formula

$$\tan 2\,\phi=\frac{2\tan\phi}{1-\tan^2\phi}=\frac{2\,x}{1-x^2}.$$

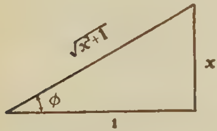


Fig. 37.

Hence $2\,\phi$ is in fact the same as the arctangent in (B), — provided $2\,\phi$ lies between $-\pi/2$ and $\pi/2$, as required for a "principal value."

Ex. II. An integration performed in two different ways gave the following results. Are they consistent?

$$(A)\quad \log\frac{\sqrt{x^2+5\,x+3}+x+\sqrt{3}}{\sqrt{x^2+5\,x+3}+x-\sqrt{3}};$$

$$(B)\quad \log\frac{\sqrt{x^2+5\,x+3}-x+\sqrt{3}}{\sqrt{x^2+5\,x+3}-x-\sqrt{3}}.$$

We form the difference $D=(A)-(B)$. Now the difference of the two logarithms is the logarithm of a quotient: viz.

the fraction in (A) divided by that in (B). Inverting the latter and multiplying, we write:

$$D=(A)-(B)=$$

$$\log\left[\frac{\sqrt{x^2+5x+3}+x+\sqrt{3}}{\sqrt{x^2+5x+3}+x-\sqrt{3}}\cdot\frac{\sqrt{x^2+5x+3}-x-\sqrt{3}}{\sqrt{x^2+5x+3}-x+\sqrt{3}}\right].$$

Multiplying out, we reduce this at once to

$$D=\log\frac{5x-2\sqrt{3}\,x}{5x+2\sqrt{3}\,x}=\log\frac{5-2\sqrt{3}}{5+2\sqrt{3}}.$$

The difference being a constant, (A) and (B) are consistent.

Remarks. (I) In canceling x in the last step above, we tacitly assume $x\neq 0$.

(II) In multiplying the long fractions just above, a short-cut is possible. The two numerators may be regarded as $(u+v)$ and $(u-v)$; and likewise the two denominators. What is u in each case? And v?

EXERCISES

1. In each of the following cases determine whether the two given functions could both be integrals of the same quantity. If so, find their constant difference.

(*a*)	y^2-7y,	$(y-1)^2-5(y-1)+6$;
(*b*)	$\log 4x^3$,	$3\log 5x$;
(*c*)	$(2x-3)^3$,	$8x^3-36x^2-54x$;
(*d*)	$\dfrac{5-2y}{y+1}$,	$\dfrac{7}{y+1}+4$;
(*e*)	$e^{4x}+e^{-4x}-\dfrac{10}{x}$,	$(e^{2x}+e^{-2x})^2+\dfrac{x-10}{x}$;
(*f*)	$\dfrac{10^x+1}{2^x}$,	$10+5^x+2^{-x}$;
(*g*)	$3+\tan^2\theta$,	$\sec^2\theta+8$;
(*h*)	$\cos 2\phi$,	$\cos^2\phi-2$;
(*i*)	$\frac{1}{4}\sin^4 t-\frac{1}{6}\sin^6 t$,	$\frac{1}{6}\cos^6 t-\frac{1}{4}\cos^4 t$;
(*j*)	$\frac{1}{4}\csc^4 t$,	$\frac{1}{2}\text{ctn}^2 t+\frac{1}{4}\text{ctn}^4 t$;
(*k*)	$\log 6x^2/k$,	$1-2\log x$;
(*l*)	$\log \text{ctn}\,\theta$,	$8-\log\tan\theta$;
(*m*)	$\sin^{-1}(3x)$,	$\pi-\cos^{-1}(3x)$;
(*n*)	$\sec^{-1}\left(\dfrac{x}{2}\right)$,	$\cos^{-1}\left(\dfrac{2}{x}\right)+\pi$.

2. Is $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$ if x is positive? If x is negative? [Take $x = -\frac{3}{5}$ and find the principal value of each function. Cf. § 37.]

3. The same as Ex. 1 for the following cases:

(*a*) $(x^2-4)^{\frac{3}{2}}+5(x^2-4)^{\frac{1}{2}}$, $\quad (x^2+1)\sqrt{x^2-4}+8$;

(*b*) $2\sqrt{2x^2+3}-\dfrac{6}{\sqrt{2x^2+3}}+\sin^2 x$, $\quad \dfrac{4x^2}{\sqrt{2x^2+3}}-\cos^2 x+5$;

(*c*) $\log(x-\sqrt{x^2-1})$, $\quad \log\dfrac{5}{x+\sqrt{x^2-1}}$;

(*d*) $2(\cos^{-1} x+\log\sec^2 x)$, $\quad \cos^{-1}(2x^2-1)-4\log\cos x$;

(*e*) $\log\dfrac{\sqrt{x^2+3x+1}+x+1}{\sqrt{x^2+3x+1}+x-1}$, $\quad \log\dfrac{\sqrt{x^2+3x+1}-x+1}{\sqrt{x^2+3x+1}-x-1}$;

(*f*) $\log\dfrac{\sqrt{3x^2+8}+x\sqrt{3}+2}{\sqrt{3x^2+8}+x\sqrt{3}-2}$, $\quad \log\dfrac{\sqrt{3x^2+8}-x\sqrt{3}+2}{\sqrt{3x^2+8}-x\sqrt{3}-2}$;

(*g*) $\log\dfrac{\sqrt{2x^2+x+4}+x\sqrt{2}+2}{\sqrt{2x^2+x+4}+x\sqrt{2}-2}$, $\quad \log\dfrac{\sqrt{2x^2+x+4}-x\sqrt{2}+2}{\sqrt{2x^2+x+4}-x\sqrt{2}-2}$.

4. Integrate $x^3(x^4+1)^2\,dx$ in two different ways, and reconcile your results.

5. The same as Ex. 4 for $\int \sin\phi\cos\phi\,d\phi$. [Regard either $\sin\phi$ or $\cos\phi$ as u.]

6. Integrate $\operatorname{ctn} u\,du$ by multiplying and dividing it by $\csc u$, and regarding $\csc u$ as v. Reconcile the result with (22), p. 101.

7. For review, write at sight the values of the following:

(*a*) $\displaystyle\int\frac{dx}{4x+5}$, $\displaystyle\int\frac{3x^2\,dx}{x^6+16}$, $\displaystyle\int\frac{2x\,dx}{\sqrt{x^4-9}}$, $\displaystyle\int\frac{4x^3\,dx}{\sqrt{x^4+1}}$, $\displaystyle\int\frac{2\,dx}{\sqrt{4-x^2}}$;

(*b*) $\displaystyle\int\tan^2 5x\,dx$, $\displaystyle\int\frac{\sec^2\theta\,d\theta}{\tan^2\theta-25}$, $\displaystyle\int\sin^3\theta\cos\theta\,d\theta$, $\displaystyle\int\csc 3x\operatorname{ctn} 3x\,dx$.

§ 67. Use of Tables. By methods to be described later, tables of integrals have been constructed. From these many of the integrals needed in practical problems can be read directly.*

Tables are printed with x as the variable, but this may have a generalized meaning, just as u heretofore.

To use any table effectively, *study its arrangement.* Then notice in each problem *what kind of function* we have to

* See pp. 492–498, or B. O. Peirce, *A Short Table of Integrals*.

integrate: Is it algebraic, trigonometric, exponential, or what? And of just what variety?

To illustrate, consider the integral $Q=\int\sqrt{9\,x^2+5}\,dx$. This is algebraic; irrational; involves x^2 under the radical; has both terms positive. It thus comes under (15), p. 493:

$$\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right),$$

provided we consider our $3\,x$ as u (or the "x" of the formula) and our 5 as a^2, and supply a necessary factor 3 to make $3\,dx$:

$$Q=\tfrac{1}{3}\int\sqrt{9\,x^2+5}\;\,3\,dx=\tfrac{1}{3}\left[\frac{3\,x}{2}\sqrt{9\,x^2+5}\;+\tfrac{5}{2}\log\left(3\,x+\sqrt{9\,x^2+5}\right)\right].$$

A required integral often involves some high power which is not shown in the table but which can be reduced, step by step, until a tabulated form is reached. Formulas for this purpose are called *Reduction Formulas.* Those given on pages 494–498 are of five kinds and permit reductions of the sorts mentioned below, except in certain cases where a denominator becomes zero and where, fortunately, the formula is not necessary.

(*A*) *Algebraic Products of the Form* $x^m(ax^n+b)^p dx$. Inspection of formulas (27)–(30), p. 494, will show that the exponent p can be lowered by one unit at each step (or raised if negative), and that the exponent m can be lowered or raised by n.

E.g., if in formula (29) we put $m=8$, $n=3$, $p=\frac{1}{2}$, $a=1$, $b=4$, we obtain, on simplifying:

$$\int x^8(x^3+4)^{\frac{1}{2}}\,dx=\tfrac{2}{21}\,x^6(x^3+4)^{\frac{3}{2}}-\tfrac{16}{7}\int x^5(x^3+4)^{\frac{1}{2}}\,dx.$$

Here the power of the binomial, $(x^3+4)^{\frac{1}{2}}$, is the same in both integrals; but the exponent of the outside power of x has been reduced from 8 to 5, *i.e.*, by 3 units (the same as the inside exponent). Applying the formula again, this time with $m=5$, would bring the integrand down to $x^2(x^3+1)^{\frac{1}{2}}\,dx$. This last is of the type $u^{\frac{1}{2}}\,du$, aside from a factor 3.

(*B*) *Somewhat Similar Forms Involving Trinomials.* These will be described later in § 100.

(*C*) *Products of x^m by Certain Transcendental Functions.* The exponent of x can be lowered by a unit or two at each step. [Verify this by examining (86), (90), (91), p. 498.]

(*D*) *A Positive Power of Any Trigonometric Function.* The exponent can be lowered by 2 units at each step. [See (60)–(65), p. 496.]

Any negative power can first be expressed as the corresponding positive power of the reciprocal function.

(*E*) *Products of the Form*: $\sin^m x \cos^n x \, dx$. Either m or n can be lowered or raised by 2 units at each step. [See (66)–(69), p. 497.] Shorter methods for some cases will be shown later.

E.g., in the case of $\int \frac{dx}{\sin^3 x \cos^5 x}$ or $\int \sin^{-3} x \cos^{-5} x \, dx$, we could raise the -3 to -1; and later raise the -5 to -3, and then to -1, obtaining thus the form

$$\int \frac{dx}{\sin x \cos x}. \tag{42}$$

The formulas will not apply further, due to zero denominators. But this last integral can be read from the table, or worked out as in Ex. 14, p. 173.

EXERCISES

1. Find the integral of $\sqrt{4x^2+7}\,dx$ in two forms: (*a*) by regarding $2x$ as u; (*b*) by factoring out the 4 from under the radical and considering $2\int\sqrt{x^2+\frac{7}{4}}\,dx$. Reconcile the results.

2. Find each of the following integrals in two forms:

(*a*) $\int \sqrt{9x^2-1}\,dx$, (*b*) $\int \sqrt{25-4x^2}\,dx$,

(*c*) $\int \frac{dx}{\sqrt{7x^2+3}}$, (*d*) $\int \frac{dx}{\sqrt{4-5x^2}}$.

3. Find each of the integrals:

(a) $\displaystyle\int\frac{dx}{(x^2-25)^{\frac{3}{2}}}$, (b) $\displaystyle\int\frac{\sqrt{2x^2+5}}{x}\,dx$,

(c) $\displaystyle\int\sqrt{\frac{5-x}{2+x}}\,dx$, (d) $\displaystyle\int\frac{dx}{x\sqrt{2-3x}}$,

(e) $\displaystyle\int\frac{x\,dx}{\sqrt{2+3x-x^2}}$, (f) $\displaystyle\int\frac{\sin 4x\,dx}{e^{5x}}$,

(g) $\displaystyle\int\frac{dx}{3+2\cos x}$, (h) $\displaystyle\int x^4\log x\,dx$.

4. Reduce and finally calculate:

(a) $\displaystyle\int x^5\sqrt{9-x^2}\,dx$, (b) $\displaystyle\int\frac{dx}{(2x^2+5)^3}$,

(c) $\displaystyle\int x^4\sin 3x\,dx$, (d) $\displaystyle\int x^3e^{2x}\,dx$,

(e) $\displaystyle\int\tan^6\theta\,d\theta$, (f) $\displaystyle\int\sec^5\phi\,d\phi$,

(g) $\displaystyle\int\sin^4\theta\cos^2\theta\,d\theta$, (h) $\displaystyle\int\frac{\sin^4\theta}{\cos^2\theta}\,d\theta$.

5. Find the numerical values of:

(a) $\displaystyle\int_0^4\sqrt{4x-x^2}\,dx$, (b) $\displaystyle\int_0^{\frac{\pi}{4}}x^2\cos x\,dx$, (c) $\displaystyle\int_0^{\frac{\pi}{2}}\sin^6\theta\,d\theta$.

6. Test the consistency of each following pair of functions as possible integrals of one quantity:

(a) $5+8x^2-2x^4+\frac{1}{6}x^6$, $10-\frac{1}{6}(4-x^2)^3$;

(b) $\log 80x^5\sqrt{x-3}-20$, $5\log x+\frac{1}{2}\log(x-3)$;

(c) $\log(\sqrt{1+x}+\sqrt{1-x})+10^{2x}$, $\frac{1}{2}\log(2+2\sqrt{1-x^2})+(10^x)^2$.

7. Integrate the following by inspection:

$\tan^{10}\theta\sec^2\theta\,d\theta$, $\sin^9\phi\cos\phi\,d\phi$, $\dfrac{x\,dx}{(x^2+4)^5}$, $\dfrac{e^x\,dx}{e^{2x}+25}$.

8. Integrate $\dfrac{(3x+5)dx}{\sqrt{24x-x^2}}$ without tables, as in § 65. Also integrate by making two fractions directly and using tables.

PART II. FURTHER APPLICATIONS

We next analyze various problems involving integration. Our object is not the study of particular scientific or technical questions for their own sake, but rather mastery of a certain

method of analysis. Such mastery can only be gained by practice, — by applying the procedure in many diverse situations.

Hence, do not regard each new application as a new principle, or each new integral as another basic formula. Rather, regard these simply as further *illustrations* of a mode of analysis.

Grasp clearly the meaning of any technical words that may be introduced; and *understand the relationships* used in the reasoning. You can then build up the same integrals again at any time, with very little use of sheer memory. (Some few results, employed extensively later on, should be definitely memorized: these are indicated in the text.)

Let us start by noting carefully the general method of analysis to be used. (Cf. *Intro.*, §§ 99, 292–95.)

§ **68. Infinitesimal Elements.** Historically the sign $\int$ was originally an S standing for "sum." Now it denotes an integral or *limit* of a sum (§ 59):

$$\int_a^b f(x)dx = \underset{\Delta x \to 0}{\mathrm{L}} [f(x_1)\Delta x + \cdots + f(x_n)\Delta x].$$

The original idea amounted to this: When Δx is extremely small (usually then written dx), the sum is nearly equal to its limit. Also, each term $f(x)\Delta x$, or $f(x)dx$, is very minute, and there are exceedingly many of them. So the integral

$$\int_a^b f(x)dx$$

was regarded as the sum of a vast number of exceedingly small terms or "infinitesimal elements," $f(x)dx$.

This rough idea is so helpful that we shall employ it constantly in setting up integrals. Any desired quantity will be regarded as the sum of many tiny elements, in each of which any troublesome variable remains momentarily constant. After writing a formula for the element, on this basis, we express the entire quantity as an integral.

To illustrate, consider a familiar example: the volume of a solid. The cross section area A_s may vary with the perpendicular distance x from some fixed point. But we take an exceedingly thin slice, — not even as thick as a soap film, — and regard it as having the same area A_s on both sides. Multiplying that area by the tiny thickness dx gives $A_s dx$ as the volume of the slice. The entire volume V is the sum, for all slices, from $x=a$ to $x=b$:

$$V=\int_a^b A_s dx. \tag{43}$$

Actually, of course, A_s changes somewhat even in the tiny distance dx; and so the volume of a slice is only *approximately* $A_s dx$ [or $A_s \Delta x$]. But V is in reality the *limit* of the sum of the terms $A_s \Delta x$; and hence (43) will give a strictly correct result, provided we now regard the integral in its proper sense, and find it by reversing the differentiation process.

Similarly in other cases: if for brevity we call a quantity a sum when it is really the limit of that sum, and if we then *integrate* (getting the limit of the sum), we shall obtain a precise result. If there is ever any doubt as to the validity of a formula set up in this rough manner, simply build up the integral by a strict argument about the limit of a sum. (Cf. § 58, Ex. I.)

§ 69. **Area between Curves.** In finding some plane areas it is not convenient to use the idea of "the area under a curve." Instead we consider a typical horizontal or vertical strip, of tiny width dy or dx, express its length and hence its area; and then "sum up," *i.e.*, integrate.

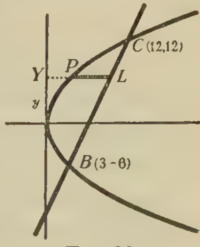


Fig. 38.

Ex. I. The parabola $y^2=12\,x$ is cut by a straight line at (12, 12) and (3, −6). Find the included area.

A horizontal strip located at any value of y has an infinitesimal width dy and a length $PL = YL - YP$. (Fig. 38.) That is, PL is x for the point L minus x for the point P. Each depends upon y and can be found from the corresponding equation of the line or parabola.

Using points B and C we get the equation of the line by (7), § 3:

$$y+6=2(x-3).$$

This gives for the x of any point L: $x_l=\frac{1}{2}(y+12)$.

$$\therefore\ PL=x_l-x_p=\frac{y+12}{2}-\frac{y^2}{12}.$$

Multiplying by the width dy gives the area of the strip; and integrating gives the entire area:

$$A=\int_{-6}^{12}\left(\frac{y+12}{2}-\frac{y^2}{12}\right)dy=\frac{1}{4}y^2+6y-\frac{1}{36}y^3\Big]_{-6}^{12}=60-(-21)=81.$$

Remark. Here it would have been troublesome to take a vertical strip. For, to the right of point B such a strip would run from the line up to the parabola, but to the left of B it would run from parabola to parabola. We should have to calculate the two parts of the area to the right and left separately by different integrals:

$$A=\int_0^3 2\,y_p\,dx+\int_3^{12}(y_p-y_l)\,dx,$$

where $\qquad y_p=\sqrt{12\,x}, \quad \text{and} \quad y_l=2\,x-12.$

§ 70. Total Load or Weight. When the load or weight resting upon a rod or plane surface is non-uniform, we find the total load by considering a portion so small that the rate of loading may be regarded as constant on it. Expressing the tiny load on the tiny element, we "sum up," *i.e.*, integrate.

Ex. I. The area between the line and parabola in Fig. 38 above carries a weight per square unit which varies thus: $w=10-.03\,y^2$. Find the total load W.

The strip shown, of length PL and width dy, is supposed so narrow as to have the same value of y throughout. Hence,

Weight per unit area $\qquad = 10-.03\,y^2$,

Number of units of area $\ =(x_l-x_p)dy=\left[\frac{y+12}{2}-\frac{y^2}{12}\right]dy.$

$\therefore$ Weight resting upon strip $=(10-.03\,y^2)\left[\frac{y+12}{2}-\frac{y^2}{12}\right]dy.$

Multiplying out and integrating, from $y=-6$ to $y=12$, gives W.

If the weight per unit area had varied with x instead of y, we should have been obliged to use the less convenient vertical strips mentioned in § 69 (Remark). If w had varied with both x and y, neither a horizontal nor a vertical strip would do. Such cases are treated in Chapter V.

Instead of a load resting upon a surface, we may wish to find the weight of a plate which has a variable density or thickness. But the process is the same.

By the way, "weight" is the *force* with which the earth pulls upon an object. The term for the quantity of matter in the object is "mass." The mass per unit area is called the "surface density," and for a rod the mass per unit length is called the "linear density."

Further suggestion. In every problem look especially for the *key idea* which must be used in expressing an element of the quantity to be found. In the examples of §§ 68–70 the key ideas are:

§ 68: Volume of thin slice = area × thickness, or $A_s\,dx$;
§ 69: Area of narrow strip = length × width, or $(x_l - x_p)dy$;
§ 70: Load on narrow strip = (weight per unit area) × (area of strip), or $(10 - .03\,y^2) \times (x_l - x_p)dy$.

EXERCISES

1. Carry out the integration in Ex. I above and find the value of W.

2. In a rectangular plate 10 in. long and 5 in. wide, the weight per sq. in. (w lb.) varies thus with the distance x in. from one end: $w = .5 - .006\,x$. Find the total weight, W lb.

3. Find the area bounded by the parabola $y^2 = 8\,x$ and the straight line $2\,x - y = 8$. [Where do they cross?]

4. The same as Ex. 3 for the following cases:

(*a*) the parabola $x^2 = 3\,y$ and the line $y = 2\,x + 9$;
(*b*) the two parabolas $y = x^2$ and $y = 10\,x - x^2$;
(*c*) $y = x^2$, and the line which cuts it at $(-1, 1)$ and $(2, 4)$;
(*d*) $y^2 = 5\,x$, and the line cutting it at $(5, -5)$ and $(20, 10)$.

5. A thin flat plate has the shape of the area bounded by $y = x^2$ and the line $y = 9$. Its surface density varies thus: $D = 2 + .06\,y$. Find its mass.

6. The same as Ex. 5 if D varies thus: $D = 3 - .2\,x^2$. [Note carefully what constitutes the length of an elementary strip.]

7. The speed v of an object varies with the time t. Express the distance traveled from $t = a$ to $t = b$: first as in § 68 by regarding v as constant for any tiny interval dt; and then again as in § 58 by a more accurate argument as to the limit of a sum.

8. In each following case express the desired quantity as an integral by using the abridged method of § 68:

(*a*) The efficiency of an electric light, or number of light units given out per hr., varies with the age of the bulb (t hr.). Express the total light given out during the first 100 hr.

(*b*) The rate of formation of vinegar from cider (R oz. per hr.) varies with t. Express the total amount formed in 20 hr.

(*c*) Atmospheric pressure changes with the elevation at a rate (R in./ ft.) which varies with E. Express the total change from $E =$ 1000 to $E = 4000$.

(*d*) The amount of solar energy received per hour by a square foot of ground varies with t. Express the total received from $t = 6$ to $t = 12$.

9. To keep the basic formulas fresh in mind, write these integrals by inspection:

$$\int \csc^2 3\,\theta\, d\theta, \qquad \int \tan 2\,x\, dx, \qquad \int \sec 4\,t \tan 4\,t\, dt, \qquad \int \sqrt{x^2+4}\; x\, dx.$$

$$\int \frac{\cos\theta\, d\theta}{1+\sin\theta}, \qquad \int \frac{6\, du}{\sqrt{u^2-25}}, \qquad \int \frac{3\, dx}{x\sqrt{x^2-25}}, \qquad \int \frac{dt}{\sqrt{8\,t-t^2}}.$$

§ 71. Torque or First Moment.

When a force F acts upon an object at a point P, it tends to produce rotation around any axis AB which does not lie in a plane with PF. (Fig. 39 *a*.) This rotational tendency is called the *torque* or *first moment* of the force. It is measured by the product Fx, where x is the "arm" or length of the common perpendicular between AB and PF.

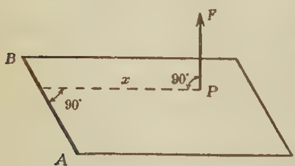


FIG. 39 *a*.

If a force does not act at a single point, but is "distributed" (that is, applied all along a rod or over a plate), its total torque can be found by integration.

Ex. I. A thin quarter-circular plate of radius 10 in. weighs .004 lb. per sq. in. Find the torque or moment of its weight about one straight side OY as an axis when the plate lies horizontally. (Fig. 39 b shows the plate in perspective. The vertical arrows represent the weights of particles in a narrow strip having a common arm x throughout.)

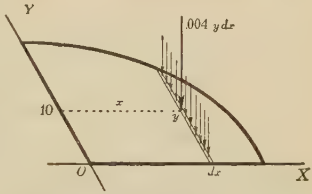


FIG. 39 b.

The weight of the strip is $.004\,ydx$; and this creates a torque $.004\,xydx$. The total is the sum, for all strips from $x=0$ to $x=10$:

$$T=\int_0^{10} .004\,xydx.$$

Remarks. (I) To work this out, we would use the fact that y varies with x, viz. $y=\sqrt{100-x^2}$. The integral can then be read from a table or found by (1), p. 87:

$$T=-\frac{.004}{3}(100-x^2)^{\frac{3}{2}}\Big]_0^{10}=\frac{4}{3}. \tag{44}$$

(II) Torque units are called pound-inches or pound-feet, etc. But foot-pounds are units of work.

§ 72. **Bending Moment.** In studying the bending of a loaded beam, the first step is to find what is called the "bending moment" (*B.M.*) at any point P. This means simply the algebraic sum of the moments or torques about P of all external forces applied to that portion of the beam to the left of P. Any torque tending to rotate that portion clockwise about P is considered positive; opposite to this, negative.

A supporting force F_a at a pier is considered as acting at a point; and its moment about P is found by multiplying F_a

by its perpendicular arm PA. For the weight of the beam itself, consisting of many particle weights, — and for any other "distributed" load, — the total moment is found by integration, as in § 71.

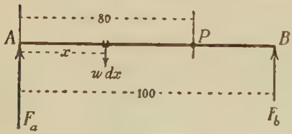


Fig. 40.

Ex. I. A beam 100 inches long rests on piers at its ends A and B. The loading (w lb. per in.) including the weight of the beam varies thus with the distance (x in.) from A:

$$w = 3 - .012\,x. \tag{45}$$

Find the $B.M.$ at a point P located 80 in. from A.

As a preliminary step we must find the supporting force F_a at A. Simply consider moments about B. Any elementary load wdx has the arm $(100-x)$, and hence the moment $(100-x)wdx$. The total moment about B due to the load is

$$M_b = \int_0^{100} (100-x)(3-.012\,x)dx = 13000. \tag{46}$$

This must be balanced by the moment of F_a about B, viz. 100 F_a.

$$\therefore\ F_a = \frac{13000}{100} = 130.$$

The $B.M.$ at P is now found similarly by considering moments about P of all forces applied to the left. The arm for F_a is now 80; and its moment 80(130) or 10400, which is considered positive by the rule above. The elementary load wdx now has the arm $80-x$, and the moment $(80-x)wdx$, except that it is to be considered negative. Summing up, we take all elements from $x=0$ to $x=80$. In all, the $B.M.$ at P is:

$$B.M. = 10400 - \int_0^{80} (80-x)wdx. \tag{47}$$

Replacing w by its given value $(3-.012\,x)$, and integrating:

$$B.M. = 10400 - [240\,x - 1.98\,x^2 + .004\,x^3]_0^{80} = 1824. \tag{48}$$

Remark. To find the bending moment at any distance X from A we could proceed likewise. But the arm for F_a would be X instead of 80, and the arm for $w\,dx$ would be $X-x$ instead of $80-x$. Also the upper limit of integration would be X instead of 80. The $B.M.$ so calculated would turn out to be:

$$B.M._X = 130\,X - 1.5\,X^2 + .002\,X^3. \tag{49}$$

This idea will be referred to later.

EXERCISES

1. In a horizontal rod 10 ft. long, the weight per foot at a distance x ft. from one end A is $w=5+.06\,x$. Find the total weight; also the torque about A.

2. In Ex. 1 find the torque about the other end. [What arm has the weight of any particle x ft. from A?]

3. A horizontal rectangular plate is 4 ft. long and 2 ft. wide. The weight per sq. ft. at a distance x ft. from one end AB is $w=5-.2\,x$. Find the torque about AB as an axis.

4. In Ex. 2, p. 119, find the torque about the end mentioned.

5. A flat horizontal plate has the shape bounded by $y=x^2$ and the line $y=4$, the unit being 1 ft. It weighs 3 lb. per sq. ft. Find the torque about its straight side.

6. The same as Ex. 5 for a semi-circular plate of radius 2 ft.

7. A beam 300 in. long rests on piers at its ends A and B. The loading is uniform, 400 lb. per in. Find the total load and its torque about A or B. Also find the supporting forces at A and B.

8. For the beam in Ex. 7 find the bending moment at a point 120 in. from A.

9. The same as Ex. 7 for a beam 200 in. long if the loading (w lb. per in.) varies thus with the distance: $w=1000-2\,x$.

10. For the beam in Ex. 9 find the $B.M.$ at $x=80$.

11. Integrate by inspection:

$$\sin^2\theta\,d\theta,\qquad \tan^2 x\,dx,\qquad (x^3+1)^{10}\,x^2\,dx,\qquad \operatorname{ctn}^3 t\csc^2 t\,dt,$$

$$\frac{3\,dx}{x^2+4},\qquad \frac{dx}{\sqrt{9-5\,x^2}},\qquad \frac{e^u\,du}{e^{2u}-1},\qquad \frac{2\,y\,dy}{\sqrt{1-y^2}}.$$

§ 73. Centroid. The center of gravity or "centroid" of an object is the point G at which the object would balance if supported there by a single force equal to its weight. To find G we consider torques about any horizontal axis.

For instance, in the flat plate of Fig. 41, let the straight sides be the X- and Y-axes, $\bar{x}$ the abscissa of G, and W the weight of the plate. Then the torque of the supporting force W about the Y-axis is $W\bar{x}$. For a balance this must equal the total torque T due to all the tiny weights of elements. That is, $W\bar{x} = T$, or

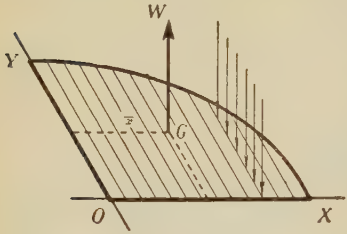


Fig. 41.

$$\bar{x} = \frac{T}{W}. \tag{50}$$

A similar equation holds for the ordinate $\bar{y}$ of G and the torque about the X-axis.

Thus the centroid is known if we can find the weight and the torques about the Y- and X-axes. These are obtainable by integration as in §§ 70–71 — at least when the surface density D is constant.* For then the weight dW of an element parallel to the Y-axis, with arm x, is known. And T and W in (50) are:

$$T = \int x\,dW, \qquad W = \int dW. \tag{51}$$

Or, since the weight dW is proportional to the mass dM, we may say that $\bar{x}$ and $\bar{y}$ are given by

$$\bar{x} = \frac{\int x\,dM}{\int dM}, \qquad \bar{y} = \frac{\int y\,dM}{\int dM}. \tag{52}$$

These symbolic formulas for $\bar{x}$ and $\bar{y}$ are used often and should be memorized. But keep in mind also the basic idea of *balancing:* viz. the total torque due to element-weights must be equal to the torque of the whole weight W, applied with a single arm $\bar{x}$ or $\bar{y}$.

* The general case where D varies is discussed in § 118.

EXERCISES

1. In a horizontal rod 20 ft. long the weight per foot, at a distance of x ft. from one end A, is $w = 2 + .06\,x$. Find the torque about A. Also locate the centroid.

2. The same as Ex. 1 for a rod 10 ft. long if $w = 2 - .004\,x^2$.

3. In a rectangular plate 12 ft. long and 5 ft. wide, the surface density D varies thus with the distance (x ft.) from one end AB: $D = 8 - .3\,x$. Locate the centroid.

4. Find the centroids for plates of the following shapes, each of constant surface density $D = k$:

(*a*) Semi-circular, of radius 10 ft.;

(*b*) Right triangular, with sides 9 ft. and 6 ft.;

(*c*) The parabolic shape in Ex. 5, p. 120;

(*d*) Quarter-elliptic with $a = 5$ and $b = 3$. [What equation has the ellipse?]

(*e*) The figure bounded by $y^2 = 2\,x$ and $y = x - 4$. [In finding $\bar{x}$, see § 69, *Remark*.]

(*f*) Check the result in (*b*) by showing without calculation that the centroid must lie on each median.

5. A plate has the shape of the area between $y = x^2$ and $y = 16$; and its surface density D varies thus: $D = 10 - \frac{1}{2}\,y$. Find $\bar{y}$.

6. A beam 100 in. long rests on piers at its ends A and B. The loading per inch at any point x in. from A is $w = 40\,x - .2\,x^2$. Find the supporting force at each pier.

7. In Ex. 6 find the $B.M.$ at a point 60 in. from A.

8. Test the consistency of the following two functions as possible integrals of the same quantity:

$$5 + \log \frac{\sqrt{x^2+4} - 2}{x}, \qquad 1 - \log \frac{\sqrt{x^2+4} + 2}{x}.$$

9. The same as Ex. 8 for these two functions:

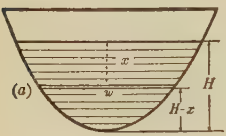
$$\operatorname{ctn}^{-1} x + \log \sqrt{\frac{x-1}{x+1}}, \qquad \pi - \tan^{-1} x - \log \sqrt{\frac{x+1}{x-1}}.$$

10. What kind of function is each of these integrals:

$$\int \frac{e^x\,dx}{\sqrt{9+e^{2x}}}, \quad \int \frac{y\,dy}{\sqrt{9-y^4}}, \quad \int \frac{\sin\theta\,d\theta}{\sqrt{9-4\cos\theta}}, \quad \int \frac{\cos\phi\,d\phi}{\sqrt{\sin^2\phi - .25}}?$$

§ 74. Center of Water Pressure. The force exerted by water against the vertical face of a dam tends to upset the

dam, *i.e.*, turn it about a horizontal axis at the bottom. The total torque is called the "Upsetting Moment" (*U.M.*). To find this, multiply the force against any tiny horizontal strip, 62.5 $xwdx$, by its arm or distance above the bottom, and then integrate. If the entire height of the dam up to the surface of the water is H, the arm for any strip is $H-x$. (Fig. 42 *a* shows a face view, and 42 *b* a side view of the dam.)



$$\therefore \quad U.M. = \int_0^H 62.5\, xw\,(H-x)\,dx. \qquad (53)$$

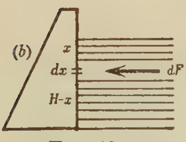


Fig. 42.

The point on a dam at which the entire force of pressure F, if applied concentrated, would produce the same moment or torque about any horizontal or vertical axis, as is produced by the actual distributed pressure, is called the Center of Pressure (*C.P.*).

Why must there be some such point?

To calculate the height $\bar{y}$ of the *C.P.* above the bottom, we simply equate the product of the total force and its arm, viz. $F\bar{y}$, to the *U.M.*, — after finding both F and *U.M.* by integration.

Remarks. (I) We may locate the *C.P.* also by finding the torques about any other horizontal axis, *e.g.*, one at the surface of the water.

(II) For any other liquid than water a different numerical factor replaces 62.5 in both F and *U.M.*, but the *C.P.* is the same.

§ 75. **Flow of Water.** If water moved through a pipe of cross section area A sq. ft. with a constant speed of V ft./sec. at all points in the pipe, a column V ft. long would pass any point in one second, and the volume or quantity passing would be

$$Q = V \cdot A. \qquad (54)$$

Actually the speed is greater at the center of the pipe than near the circumference, and varies with the distance x from

the center. Hence, to express the total flow per second we would consider a very narrow ring of width dx at a distance x feet from the center, and regard the speed as constant at all points on this ring. (See Ex. 6, p. 128.)

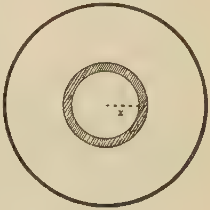


FIG. 43.

Flow through large orifices. If, instead of a pipe, we consider a large opening in a dam, the speed of flow will be virtually the same at all points on the same horizontal level. But it will vary with the "head," or vertical distance (x ft.) below the still upper surface some distance back of the dam. In fact, very closely,

$$v = K\sqrt{x},$$

where K is a constant depending on the shape of orifice.

Denoting by w the width of the opening at any depth or head x, the flow per second through the entire opening is

$$Q = \int_a^b K\sqrt{x}\, w\, dx, \tag{55}$$

where a and b are the distances of the top and bottom of the orifice below the still upper surface of the water.

The foregoing principles hold also for other liquids besides water, — with slightly different constants.

§ **76. Work Done by a Gas.** As a gas expands, the pressure which it exerts (p lb. per sq. in.) varies with its volume (v cu. in.). The work which it does is simply

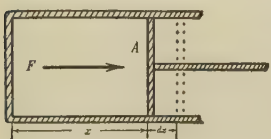


FIG. 44.

$$W = \int p\, dv. \tag{56}$$

We shall at present establish this formula only in the case of expansion within a cylinder. (Fig. 44.) The force F against the piston equals the pressure p multiplied by the area of the

piston (A sq. in.). The work done while the piston moves dx inches is

$$dW = F\,dx = (p\,A)dx.$$

But $$A\,dx = dv,$$

whence $$dW = p\,dv. \tag{57}$$

A similar argument in more general form will show that formula (56) is correct in all cases. Thus, for gases, (56) takes the place of the more general work formula, $W = \int F\,dx$.

EXERCISES

1. In what respects is the $C.P.$ for a dam like the centroid for a flat plate? Wherein different?

2. Show in detail the steps by which equation (55) is set up. [Take a narrow horizontal strip across the opening and use (54).]

3. For a vertical dam 10 ft. high the width at any depth x feet below the surface of the water is $w = 400 - x^2$. Calculate the total force of water pressure against the dam. Also find the $U.M.$ and the $C.P.$

4. Like Ex. 3 if $w = 300 - x^2$.

5. A triangular gate in a dam is 2 ft. tall. Its top (6 ft. across) is 10 ft. below the surface of the water. Find the total force against the gate; also the $C.P.$

6. In a circular water pipe of radius 10 in. the velocity of flow (V in./sec.) x in. from the center is $V = 100 - .6\,x^2$. Find the total flow per second past any point.

7. The same as Ex. 6 if $V = 100 - x^2$.

8. A triangular notch in a dam is 3 ft. deep and 4 ft. wide at the top. The surface of still water some distance back of the dam is 1 ft. higher than the top of the notch or dam. Find the amount of flow per second, if $K = 5$ in (55).

9. For a certain gas $pv = 500$. Find the work done in expanding, from $v = 8$ to $v = 20$. ["Isothermal expansion."]

10. If a gas expands so that $pv^{\frac{4}{3}} = 8000$ constantly, find the work done, from $v = 8$ to $v = 27$. ["Adiabatic expansion."]

11. Test the consistency of these two functions as possible integrals of the same quantity:

$$2\cos^{-1} x + \log\sqrt{x^2 + \frac{1}{x^2}}, \quad \sin^{-1}(2\,x\sqrt{1 - x^2}) + \tfrac{1}{2}\log(x^4 + 1) - \log x.$$

§ 77. **Moment of Inertia.** An unbalanced force F applied to an object produces rotation about any axis AB which is not in a plane with the direction of F. The angular acceleration imparted is proportional to the torque or moment of F about AB. (Fig. 45.)

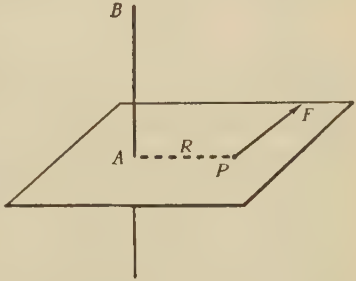


FIG. 45.

The torque which must be applied *per unit angular acceleration* imparted, is called the "moment of inertia" of the body with respect to the axis in question. It is denoted by I.

In the case of a mere particle P, of mass dm, at a distance R from the axis of rotation, the moment of inertia is

$$I = R^2\, dm. \tag{58}$$

Proof. Let F act at right angles to AP, so that its torque is FR. And let a be the acceleration produced by F. Then, if the units are suitably chosen, the force F equals the acceleration a times the mass:

$$F = a\, dm. \tag{59}$$

Moreover, a equals the angular acceleration (α radians per sec. gained per sec.) times the perpendicular radius R.* Hence

$$F = R\alpha\, dm. \tag{60}$$

Thus the torque per unit angular acceleration ($FR \div \alpha$) is:

$$I = R^2\, dm. \qquad \text{Q.E.D.}$$

§ 78. ***I* for Rods and Plates.** Equation (58), which should be memorized, gives the moment of inertia for a *particle*. But evidently it holds also for a thin rod parallel to the axis AB, if dm is the mass of the rod. To calculate I for any other rod, we should have to regard it as composed of particles; and integrate.

* Cf. *Introduction*, §§ 249–251.

For a flat *plate* and an axis in or parallel to its plane, we consider the plate as composed of narrow strips parallel to the axis. For an axis perpendicular or otherwise inclined to the plate, we should have to resort to particles. (Chapter V.)

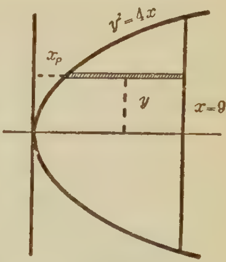


FIG. 46.

Ex. I. A flat plate whose surface density is .003 gm. per sq. cm. has the shape of the figure bounded by the parabola $y^2=4x$ and the line $x=9$. Find its moment of inertia with respect to its axis of symmetry.

We must use a narrow horizontal strip whose particles are all at common distance y from the axis. The length of a strip is $(9-x_p)$, its mass $.003(9-x_p)dy$, and its moment of inertia

$$.003(9-x_p)y^2\,dy.$$

The total for the plate is the sum of these tiny elements for all strips from $y=-6$ to $y=6$:

$$I=\int_{-6}^{6}.003(9-x_p)y^2\,dy. \tag{61}$$

Before integrating we must replace x_p by its value at any point of the given parabola, viz. $x_p=y^2/4$. Also, since the integrand has the same value for positive and negative values of y, we may change the limits and take twice the integral from 0 to 6.

Thus we find

$$I=.006\left[3\,y^3-\frac{y^5}{20}\right]_0^6=1.5552.$$

(The physical unit is called a *gm.-cm*^{2}.)

Remarks. (I) A torque five times as great as I, with respect to the axis in question, would give the plate an angular acceleration of 5 radians per sec. gained per sec.

(II) We are treating a thin plate as if its particles were all exactly in one plane. In a practical sense, a plate is "thin" if the error arising from this assumption is too small to cause any trouble. An exact method, taking account of all three dimensions of a solid, will be shown in § 139.

A mere *area*, of course, has no mass, weight, moment, or center of gravity in the physical sense. But by analogy any element of area dA times its distance R from an axis is called the moment (or first moment) of dA with respect to that axis. Likewise R^2dA is called the second moment, R^3dA the third moment, etc. The respective integrals give the successive total "moments of area."

The "centroid of an area," similarly defined by $\bar{x} = \int x dA \div A$, etc., corresponds to the centroid for a thin plate of constant surface density D.

§ 79. **Radius of Gyration.** The distance from a specified axis of rotation at which the entire mass of a body, if concentrated as a particle, would have the same moment of inertia as it actually has while distributed, is called the *radius of gyration*. This is denoted by K.

If the total mass is M, the moment of inertia for the entire concentrated mass would be K^2M. Since this must equal the actual I, we have $K^2M = I$, or

$$K = \sqrt{\frac{I}{M}}. \tag{62}$$

This is important in studying rotation and some oscillatory motions.

EXERCISES

1. A straight rod 40 cm. long has a constant linear density, $D = k$. Find its moment of inertia and radius of gyration with respect to axes perpendicular to it as follows:

(*a*) Through one end; (*b*) Meeting it 10 cm. from one end.

2. (*a*), (*b*). The same as Ex. 1, if D varies thus with the distance (x cm.) from the end in question: $D = 5 - .06\,x$.

3. A flat rectangular plate 20 cm. long and 12 cm. wide has a constant surface density, $D=k$. Find its I and K with respect to axes in its plane as follows:

(*a*) Either of the longer sides; also either end;
(*b*) Parallel to either long side, through the center;
(*c*) External, 5 cm. from either long side.

4. The same as Ex. 3 (*c*) if D varies thus with the distance (x cm.) from the chosen longer side: $D=4+.2\,x$. [What is then the arm R, for any elementary lengthwise strip?]

5. Find I and K for a flat plate of constant surface density, $D=k$, for each following shape and axis:

Shape	*Axis*
(*a*) Right triangle, legs 8 and 5;	shorter leg;
(*b*) Same as (*a*);	longer leg;
(*c*) Quarter-circle, radius 6;	either straight side;
(*d*) Quarter-ellipse, $a=10$, $b=6$;	longer straight side;
(*e*) Circle of radius 10;	a diameter.

6. The same as Ex. 5 (*e*) for an axis perpendicular to the plate at its center. [Hint: What shape of strip is possible, all particles to be equidistant from the axis? Cf. Fig. 43, p. 127.]

7. Recalculate I in Ex. 5 (*c*) by expressing y, dx, etc., in terms of a parametric angle ϕ before integrating. [Cf. Fig. 16, p. 25.]

8. A flat plate has the shape of the area bounded by $y=x^2$ and $y=4$. Its surface density varies thus: $D=10+y$. Find its moment of inertia about a line 2 units below the X-axis.

9. Find the centroid of the plate in Ex. 8.

10. Find the third moment of area of a semi-circle of radius a, about its straight side.

11. Find the second moment, about the Y-axis, of the area bounded by $y=\frac{1}{3}\,x^2$ and $y=4\,x-x^2$. [Plot and notice the shape.]

12. Write at sight the values of the integrals:

$$\int\sqrt{5\,x}\,dx,\qquad \int\sin^6 2\,\theta\cos 2\,\theta\,d\theta,\qquad \int 10^{-5x}dx,\qquad \int\operatorname{ctn}\theta\,d\theta.$$

§ 80. Volumes of Revolution. Consider a plane area A and a line L in its plane. If A be revolved about L as an axis, out of the given plane around through space, it will trace out or "generate" some volume. The latter can be

calculated by a single integration in two different ways, — as will now be illustrated.

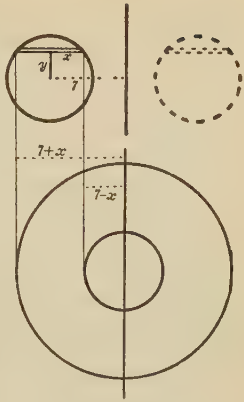


FIG. 47.

Ex. I. A circle of radius 4 in. is revolved around a vertical line in its plane 7 in. from its center. Find the volume generated.

The solid traced out resembles a doughnut, or anchor ring, or automobile tire.

(*A*) *First Method.* Any horizontal strip of the circle ($2\,x\,dy$ in Fig. 47) generates a thin flat plate whose surface is a ring between two circles, of radii $R=(7+x)$ and $r=(7-x)$. This elementary volume is then $(\pi R^2-\pi r^2)dy$, which reduces to $28\,\pi x\,dy$.

Further $\quad x=\sqrt{16-y^2}$. Hence we find

$$V=2\int_0^4 28\,\pi\sqrt{16-y^2}\,dy. \tag{63}$$

Worked out by formula (22), p. 493, this gives

$$V=28\,\pi\left[y\sqrt{16-y^2}+16\sin^{-1}\frac{y}{4}\right]_0^4=224\,\pi^2.$$

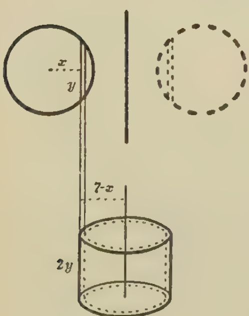


FIG. 48.

(*B*) *Second Method.* Any vertical strip of the revolved circle ($2\,y\,dx$ in Fig. 48) generates a thin cylindrical shell, of height $2\,y$, thickness dx, and radius $r=(7-x)$.

This value of the radius is correct even for strips to the left of the center, if we consider x as negative there. For $(7-x)$ is then greater than 7.

The elementary volume is now $2\,\pi r(2\,y)dx$. This can be expressed

in terms of x alone; and we find for the entire volume, from $x=-4$ to $x=4$:

$$V=4\pi\int_{-4}^{4}(7-x)\sqrt{16-x^2}\,dx. \tag{64}$$

Making two integrals of this, we obtain

$$V=\left[14\pi\left(x\sqrt{16-x^2}+16\sin^{-1}\frac{x}{4}\right)+\frac{4\pi}{3}(16-x^2)^{\frac{3}{2}}\right]_{-4}^{4}=224\pi^2.$$

Remarks. (I) In substituting -4, we take $\sin^{-1}(-1)$ as $-\pi/2$ rather than $3\pi/2$, because of our agreement as to the "principal value" of the arcsine (§ 36), upon which the differentiation formulas were based.

(II) It would not be correct in (B) above to calculate V from $x=0$ to $x=4$ and then double. Why would this be wrong, obviously?

§ 81. Length of a Curve. In the *Introduction*, §§ 293–94, we obtained expressions for the length s of any curved arc. Let us summarize briefly.

(A) *In Rectangular Coördinates.* Regard a tiny arc ds as straight, and as the hypotenuse of a right triangle with legs dx and dy. (Fig. 49.) Then

$$ds=\sqrt{dx^2+dy^2}. \tag{65}$$

Factoring out dx^2 or dy^2 under the radical:

$$\boldsymbol{ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,} \tag{66}$$

or

$$ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy. \tag{67}$$

[Memorize (66), but be able also to derive (65)–(67) as above.]

By finding dy/dx in terms of x, or dx/dy in terms of y, from the equation of the given curve, and substituting in (66) or (67), we can find the length s on integrating.

When given parametric equations, x and y in terms of ϕ, simply find dx and dy in terms of $d\phi$; then use (65) and integrate.

(*B*) *In Polar Coördinates.* Regard a tiny arc ds as the hypotenuse of a right triangle whose legs are dr and $r\,d\theta$, the latter being a tiny circular arc with radius r and central angle $d\theta$. Then $ds=\sqrt{dr^2+r^2d\theta^2}$. Factoring:

$$ds=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta, \qquad (68)$$

or

$$ds=\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr. \qquad (69)$$

From the equation of the given curve, we should find r and $dr/d\theta$ in terms of θ, or else $d\theta/dr$ in terms of r, substitute in one of equations (68), (69), and then integrate.

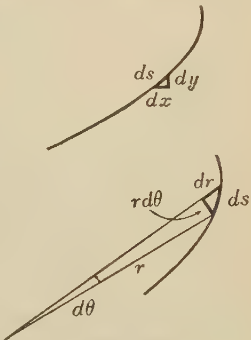


FIG. 49.

§ 82. **Curved Rods or Wires.** To find a torque, or moment of inertia, etc., for a very fine curved rod or wire, we proceed as in the case of a straight rod or flat plate. But each " particle " is now of length ds and mass $dM=D\,ds$, where D is the linear density, or mass per unit length. After expressing the torque, or moment of inertia, etc., for dM, we sum up by integration.

Ex. I. For the parabola $y=x^2$, let us express by definite integrals the several quantities (*A*)–(*C*) mentioned below:

(*A*) *Length of the curve, from $x=0$ to $x=2$.* Since $dy/dx=2\,x$, (66) gives

$$s=\int_0^2\sqrt{1+4\,x^2}\,dx. \qquad (70)$$

(*B*) *Coördinates $(\bar{x}, \bar{y})$ of the centroid of a fine wire* having the shape of the arc in (*A*), and having a constant linear density k. By (52), p. 124:

$$\bar{x}=\frac{\int x\,dM}{\int dM}, \qquad \bar{y}=\frac{\int y\,dM}{\int dM}.$$

But $dM = k\,ds$ and $M = ks$, given by (70); moreover, $y = x^2$.

$$\therefore \qquad \bar{x} = \frac{\int_0^2 x\sqrt{1+4\,x^2}\,dx}{s}, \quad \bar{y} = \frac{\int_0^2 x^2\sqrt{1+4\,x^2}\,dx}{s}. \qquad (71)$$

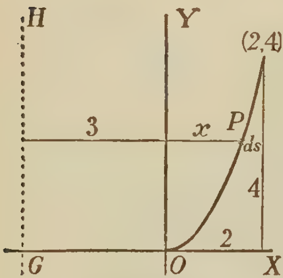


Fig. 50.

For evidently k would cancel out, above and below.

(*C*) *Moment of inertia of the same wire* with respect to an axis GH which lies 3 units to the left of the Y-axis. (Fig. 50.)

Any particle P, of mass $k\,ds$, is at a distance $(x+3)$ from GH. Hence R^2dM is $(x+3)^2\,k\,ds$.

$$\therefore \qquad I = k\int_0^2 (x+3)^2\sqrt{1+4\,x^2}\,dx. \qquad (72)$$

EXERCISES

In Ex. 1–6, find the volume generated by revolving each area in question about an axis in its plane, as specified.

1. A semi-circle of radius 5 in., about its flat side. Calculate in two ways, and also check by elementary geometry.

2. A semi-circle of radius 10 in., about an exterior axis 2 in. from the flat side. Calculate in two ways.

3. A semi-ellipse, with $a = 5$ and $b = 3$, about its flat side $2\,a$.

4. A circle of radius 3 in., about a line 15 in. from the center.

5. The area bounded by the parabola $y = x^2$ and the line $y = 9$:

(*a*) About the line $y = -9$; (*b*) About the line $x = 10$.

6. A right triangle, with legs 5 in. and 10 in., about an exterior line 3 in. from the longer leg.

7. Find the volume generated by revolving about the X-axis the area under each of the following curves, for the interval mentioned. [In (*b*) and (*c*) express each y, dx, etc., in terms of ϕ and $d\phi$.]

(*a*) One arch of the sine curve, $y = \sin x$;

(*b*) One arch of the cycloid, $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$;

(*c*) The hypocycloid, $x = a\cos^3\phi$, $y = a\sin^3\phi$, from $\phi = 0$ to $\phi = \frac{\pi}{2}$.

8. Find the length of arc of each following curve for the interval indicated:

(*a*) Semi-cubical parabola, $y=\frac{2}{3}x^{\frac{3}{2}}$, $x=0$ to 8;
(*b*) Quarter-circle, radius 10, by (66) and check;
(*c*) Spiral of Archimedes, $r=a\theta$, $\theta=0$ to θ_1;
(*d*) Equiangular Spiral, $r=e^{a\theta}$, $\theta=0$ to θ_1;
(*e*) Comet's parabolic orbit, $r=a\sec^2\left(\frac{\theta}{2}\right)$, $\theta=0$ to θ_1.

9. A rod of constant linear density, $D=k$, has the shape of the parabola $y=\frac{1}{2}x^2$, $x=0$ to $x=3$. Find

(*a*) Its mass; (*b*) Its moment of inertia about the Y-axis;
(*c*) Coördinates $(\bar{x}, \bar{y})$ of its centroid.

10. The same as Ex. 9 if the rod has the shape of the hypocycloid $x=10\cos^3\phi$, $y=10\sin^3\phi$.

11. Find the moments of inertia for the following flat plates, of constant surface density:

(*a*) Circular, of radius 10 cm., about a tangent line;
(*b*) Quarter-elliptical, with $a=10$ and $b=6$, about the side b.

§ 83. Surfaces of Revolution. If a plane curve be revolved about an axis lying in its plane, it will generate a curved surface. Any tiny arc ds, at a distance R from the axis in question, will generate a narrow band or belt around the surface, of length $2\pi R$, width ds, and area $2\pi R\,ds$.

To find the area S of the whole surface, we express both R and ds in terms of one variable, x or y or ϕ, etc.; and integrate. The following example illustrates this and several related problems.

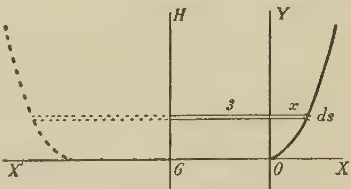


FIG. 51.

Ex. I. An arc of the parabola $y=x^2$, from $x=0$ to $x=2$, is revolved about a vertical line GH, 3 units to the left of the Y-axis. (Fig. 51.) Express by definite integrals the quantities in (A)–(C) below.

(*A*) *Area S of the surface generated.* A narrow band, of width ds has the area $2\pi R\,ds$, or $2\pi(x+3)\sqrt{1+4x^2}\,dx$. [Cf. (*C*), § 82.]

$$\therefore \qquad S=2\pi\int_0^2 (x+3)\sqrt{1+4x^2}\,dx. \tag{73}$$

(*B*) *Moment of inertia, with respect to GH*, of a thin shell having the shape of the surface in (*A*), and having a constant surface density k.

A narrow band has the mass $dM=k(2\pi R\,ds)$, and all its particles are at a distance $R=(x+3)$ from the axis GH.

$$\therefore \qquad I=\int R^2 dM=2\pi k\int_0^2 (x+3)^3\sqrt{1+4x^2}\,dx. \tag{74}$$

(*C*) *Ordinate $\bar{y}$ of the centroid of the thin shell in* (*B*). [By symmetry, the centroid must lie somewhere on GH.]

If we hold the plane of the paper horizontal, so that the plane of a band is vertical, the weight of every particle of the band will act at right angles to the paper; and will have an arm y with respect to the X-axis. Hence, balancing moments about XX' will give for $\bar{y}$, as in the case of a flat plate:

$$\bar{y}=\frac{\int y\,dM}{\int dM}. \tag{75}$$

Since $dM=k(2\pi R\,ds)=2\pi k(x+3)\sqrt{1+4x^2}\,dx$, as above, and $y=x^2$, we have

$$\bar{y}=\frac{\int_0^2 2\pi k x^2(x+3)\sqrt{1+4x^2}\,dx}{\int_0^2 2\pi k(x+3)\sqrt{1+4x^2}\,dx}. \tag{76}$$

EXERCISES

In Ex. 1–5, find the area generated by revolving each specified curve about the axis mentioned.

1. The following curves about the X-axis, $x=0$ to $x=2$:

(*a*) $y=x^3$, (*b*) $y=x^2$, (*c*) $y=x$, (*d*) $x^2+y^2=4$.

Check your answers to (*c*) and (*d*) by elementary geometry.

2. (*a*)–(*d*). The same curved arcs in Ex. 1, about the Y-axis.

3. The first quadrant of the ellipse $b^2x^2+a^2y^2=a^2b^2$:

(*a*) About the X-axis, (*b*) About the Y-axis.

4. The first quadrant of the hypocycloid $x=2\cos^3\phi$, $y=2\sin^3\phi$, about the X-axis.

5. The straight line through (2, 8) and (6, 0), about the Y-axis.

6. (*a*) From the answer to Ex. 3 (*a*), what is the surface area of a prolate spheroid whose longest and shortest diameters are 10 in. and 8 in.?

(*b*) The same for an oblate spheroid by Ex. 3 (*b*).

7. A pier 10 ft. high has a circular base and top, of radii 5 ft. and 3 ft., and is a portion of a cone. Find the area of its curved surface by using the idea of Ex. 5.

8. Find the volume of the pier in Ex. 7.

9. An arc of the circle $x^2+y^2=100$ from $y=0$ to $y=6$ is revolved about the Y-axis. Find the area of the spherical zone so generated.

10. Find the area of the surface of a torus generated by revolving a circle of radius 3 in. about an axis 10 in. from the center.

11. The reflector of a headlight has the shape generated by revolving $y=\frac{1}{8}x^2$ about the Y-axis, from $x=0$ to $x=4$. Find the area of its surface.

12. A reservoir has a flat circular base, of radius 20 ft. The curved side wall has the shape generated by revolving $y=x^2$ from $x=0$ to $x=4$ about a vertical line 20 ft. to the left of the Y-axis. Find

(*a*) The area of the wall; (*b*) The centroid of that area;

(*c*) The volume of water the reservoir can hold;

(*d*) The work required to pump such a volume of water to a level 4 ft. above the top of the reservoir, ignoring frictional losses.

13. Find the volume generated by revolving about the X-axis a triangle whose sides are: $y=x+1$, $y=2x$, $y=8$. (Plot the triangle.)

14. (*a*) Find the second moment about the X-axis, of the area bounded by $y=x^2$ and $y=x+6$. (*b*) Find the volume which would be generated by revolving the same area about the X-axis.

§ **84. Further Applications.** We shall conclude the present chapter by mentioning several more applications of integration. These are given merely for further illustration and practice; and the special principles mentioned need not be memorized. We shall consider some physical applications

first; then some from actuarial science and economics. (§§ 85–86.)

(*A*) *Illumination on the Retina.* Different colors of light, refracted through the eye, produce at any point on the retina varying illuminations. Let H denote the varying intensity per unit range in the index of refraction n. Rays within any interval dn produce an illumination $H\,dn$, and the total for all rays between n_1 and n_2 is

$$I = \int_{n_1}^{n_2} H\,dn. \tag{77}$$

(*B*) *Magnetic Field Strength.* An electric current flowing along a wire creates a magnetic field nearby; *i.e.*, it exerts force upon any magnetic pole in the vicinity. With suitably chosen units, if I is the current strength, the current flowing in any tiny length of wire ds (at P in Fig. 52 *a*) exerts upon a unit pole at O an elementary force equal to

$$\frac{I\,ds}{\overline{OP}^2}\sin\phi. \tag{78}$$

If O and the entire wire lie in a single plane, the total force, or "strength of the field" at O, is found by integration, taking all elements ds along the wire.

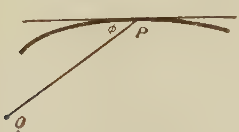


FIG. 52 *a*.

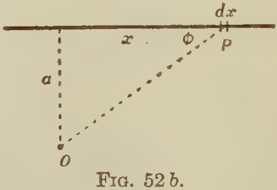


FIG. 52 *b*.

EX. I. Find the strength H of the field at a point O located a cm. from a straight wire which carries any constant current I. (Fig. 52 *b*.)

Here $ds = dx$, $\overline{OP}^2 = x^2 + a^2$, $\sin\phi = a/\sqrt{x^2+a^2}$. Using these values in (78) and integrating, we get for any length of wire from $x = b$ to $x = c$:

$$H = \int_b^c \frac{Ia\,dx}{(x^2+a^2)^{\frac{3}{2}}}. \tag{79}$$

(The direction of the force at O, by the way, is perpendicular to the plane of the paper.)

(*C*) *Kinetic Energy.* The kinetic energy (*K.E.*) of a moving object is the amount of work needed to stop the object, or to give it the present speed, starting from rest. If the instantaneous speed v is the same for all points in the object, and if m is the mass, F the tangential force and a the acceleration (both measured along the path), then the work done in a short distance dx is

$$dW = F dx = madx.$$

But $a = \dfrac{dv}{dt}$ and $dx = vdt$; whence $dW = m\left(\dfrac{dv}{dt}\right) v\, dt = mv\, dv.$

The total work done in raising the speed from zero to V is

$$K.E. = W = \int_0^V mv\, dv = \tfrac{1}{2}\, mV^2.$$

If different particles or elements in an object have different speeds, we take the kinetic energy of any one element as $\frac{1}{2} v^2\, dm$; and then integrate again. (Cf. Ex. 5, p. 142.)

(*D*) *Heat Produced by Friction.* In moving an object, a certain amount of work is wasted in overcoming friction. This is converted into heat. If the frictional force is F, the work lost per unit of time is $(F\, dx) \div dt$ or

$$Fv, \tag{80}$$

where v is the speed.

In case F or v varies with t, — or from point to point in the object, — integration is used to find the total work lost in heat during a long time, or for the whole body per unit of time, as the case may be.

We consider only cases in which the kinetic friction for a particle or element follows one of these laws:

(1) *Dry surfaces:* F equals a constant f times the force N which presses the surfaces together.

(2) *Lubricated surfaces*, separated by a lubricating film of some constant thickness: $F = kA\, v$. Here k is a constant, A the moving area of lubricated contact, and v the speed.

Ex. II. Find the total frictional loss per second for a lubricated flat circular plate of radius a, resting on a flat surface and rotating with an angular speed of ω rad./sec.

For a ring of width dx (Fig. 53), the speed v equals ω times the radial distance x: $v = \omega x$. The area of the ring is $2\pi x\,dx$. Hence the friction on the ring is

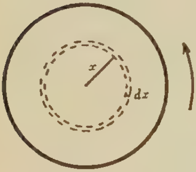


FIG. 53.

$$F = k(2\pi x\,dx)(\omega x) = 2\pi k\omega x^2\,dx.$$

By (80) the work wasted per sec. equals Fv or

$$2\pi k\omega^2 x^3\,dx. \tag{81}$$

Hence the total loss per second for the whole plate is:

$$H = 2\pi k\omega^2 \int_0^a x^3\,dx = \tfrac{1}{2}\pi k\omega^2 a^4.$$

EXERCISES

1. Find I in (77) if $H = B/(n-N)^2$ where B and N may be regarded as known constants.

2. Find H in (79) if $I = 4$, $a = 2$, $b = -100$, $c = 100$.

3. A circular coil has N turns of radius a, virtually in one plane, and carries an electric current I. If suspended in a uniform magnetic field of strength H, with the plane of the coil parallel to the direction of the field (Fig. 54), any element ds will be acted upon by a force *perpendicular to the plane* at ds and of intensity $NIH \sin\theta\,ds$. Find the total torque tending to turn the plane of the coil about the line AB. [Hints: What *arm* does the elementary force have? Express ds in terms of $d\theta$.]

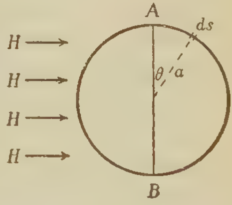


FIG. 54.

4. The current in the coil of Ex. 3 itself creates a field, whose strength at P in Fig. 55 is $2\pi a^2 NI \div (x^2 + a^2)^{\frac{3}{2}}$. Find the strength of field H at one end of the axis of a long cylindrical coil which has n turns per unit axial length, the total length being L units.

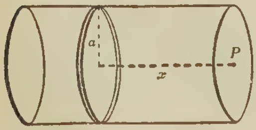


FIG. 55.

5. Find the kinetic energy of the rotating plate in Ex. II above, if the mass per unit area is a constant D.

6. Find the kinetic energy of a rod of length L and constant linear density D, which rotates with angular speed ω rad./sec. about an axis perpendicular to it at one end.

7. If the plate in Ex. II, p. 142, were not lubricated, but had a constant weight D per unit area, find the frictional loss per second.

8. A flat lubricated plate of area A is drawn straight along a plane surface with an acceleration of 4 ft./sec.^{2} What is its speed after t sec.? What frictional loss from $t=5$ to $t=10$?

9. A certain flat dry plate of length a, width b, and weight D per unit area, is of varying roughness. The coefficient f in (1), p. 141, varies thus with the distance x from one end of the plate: $f=.2+.004\,x$. Find the total friction at any instant. Hence what frictional loss in moving the plate a distance y?

10. The amount of moisture (E lbs.) evaporated hourly by a steady horizontal wind, from a strip of water l ft. long in the direction of the wind and w ft. wide, is $kw\sqrt{l}$, where k is some constant. Find E for a mill pond, if the surface has the shape of the area bounded by the parabola $y=x^2$ and the line $y=625$, and if the wind blows parallel to the X-axis.

11. Find E for the pond in Ex. 10 if the wind blows parallel to the Y-axis.

§ 85. Continuous Investment. Money is invested or paid out by insurance companies and some other corporations at such frequent intervals that actuarial calculations concerning the funds can be simplified by regarding the payments as *continuous*. The examples below will show the idea; but let us first recall some basic formulas. (*Intro.*, §§ 145, 311.)

When interest is compounded or converted annually at the rate r, the amount after n yr. for a single payment P is

$$A=P(1+r)^n; \tag{82}$$

and the "present value" n years in advance is

$$V=\frac{P}{(1+r)^n}. \tag{83}$$

These formulas are often used with fractional values of n instead of computing simple interest for a fraction of a year. For ordinary rates, they give nearly the same result.

The number of years is often denoted by t instead of n, and the interest rate by i instead of r.

Ex. I. A total of $\$P$ is to be invested, in many tiny installments spread uniformly through one year, beginning now. What will the amount be 10 yr. hence, at 6%?

During a short fraction of a year, dt, the sum deposited will be Pdt. It will remain at interest for some time t yr., which will range from 10 yr. to 9 yr. for different deposits. The amount for the one deposit mentioned will be $(Pdt)(1.06)^t$, and for all:

$$A = P\int_9^{10}(1.06)^t\,dt = \frac{P[1.06^{10} - 1.06^9]}{\log_e 1.06}. \tag{84}$$

§ 86. Applications in Economics. Many basic ideas of Economics are essentially mathematical, and are expressed most concisely in terms of derivatives or integrals. The expressions are not commonly used for purposes of calculation, but rather for brief accurate statement. A few non-technical illustrations follow.

(A) *Total Production at a Varying Rate.* A man produces or creates wealth at a varying rate ($\$R_p$ per yr.) and consumes it at another rate ($\$R_c$ per yr.). His net or excess production in a short time dt at any age t is $(R_p - R_c)dt$, and his net total during a lifetime of L yr. is

$$N = \int_0^L (R_p - R_c)\,dt. \tag{85}$$

In childhood, where $R_p < R_c$, the integrand is negative. The negative sum or integral during those years partially cancels the positive sum for the adult years.

Graphically speaking: if two curves be plotted with R_p and R_c as ordinates, and with t as the common abscissa, the area between the curves will represent N, provided we subtract any area where $R_p < R_c$ from the rest. (Cf. § 60.)

(B) *Total Utility.* By the utility of a commodity for any person is meant the value which it has for him in satisfying his desires, affording pleasure, preventing pain, etc. The "marginal utility" m is the value per unit, of a small additional amount, to a person already having any quantity x.

Usually m decreases rapidly as x becomes large. (Fig. 56.) A small addition dx has the value $m\,dx$. The total value of a quantity X is

$$U = \int_0^X m\,dx. \tag{86}$$

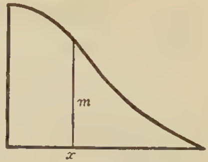


FIG. 56.

(*C*) *Contingent Values.* If a contract is to pay $10,000 twenty years hence, its present value (discounting at 6%) is about $3118. But suppose the money is to be paid only if some event occurs, the probability of which is ½. The value of the "expectation" is then only half of $3118, or $1559.

If a vast number of such contracts for similar events were bought at $1559 each, the buyer would neither win nor lose substantially, but would realize his expected 6% return; for in virtually half of the cases he would collect the full sum, and in the other half nothing.

In general, the value of an expectation is the value of the amount receivable, multiplied by the probability p of the event upon which the payment is contingent.

Suppose now that a boy just born will consume and produce wealth in accordance with (*A*) above while he lives. Discounting at 6%, the present value of his net contribution $(R_p - R_c)dt$ at age t, if he lives to make it, is by (83):

$$\frac{(R_p - R_c)dt}{(1.06)^t}. \tag{87}$$

Let the probability of his living to age t be P_t, varying with t. Then the present value of the expectation of his elementary net contribution mentioned above is P_t times the fraction in (87). And the total probable present value of his net contribution during life is

$$V = \frac{\int_0^L P_t(R_p - R_c)dt}{(1.06)^t}. \tag{88}$$

This is his probable "economic value" at the present time.

(*D*) *Historical Conjecture.* Suppose it is known that two events X and Y occurred in the period 201–225 A.D., inclusive, and that X preceded Y. And nothing else is known as to their dates. What is the probability that Y fell within the years 216–220?

If Y fell in a tiny interval dt, t yr. after the beginning of A.D. 201, X had a possible range of t yr. out of 25 yr., and Y a range of dt yr. (Fig. 57.) The number of possible ways of dating X in that range would be proportional to t; and of dating Y, proportional to dt. Hence the number of ways of so placing X and Y would be $kt\,dt$, where k is some constant factor of proportionality. And the total number of ways with Y falling between the end of 215 and the end of 220, is

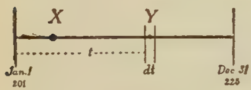


FIG. 57.

$$k\int_{15}^{20} t\,dt,$$

while the total number of ways for Y to fall anywhere in the twenty-five years is a like integral from $t=0$ to $t=25$. According, then, to the definition of probability (*Intro.*, § 334), the required chance is

$$p=\frac{k\displaystyle\int_{15}^{20} t\,dt}{k\displaystyle\int_{0}^{25} t\,dt}=\frac{\frac{1}{2}(400-225)}{\frac{1}{2}(625)}=\frac{7}{25}.$$

The chance that Y occurred within the specified five years is to the chance that it occurred elsewhere in the 25 yr. as 7 is to 18.

EXERCISES

1. (*a*) In Fig. 56 what could represent the total utility U of an amount X?

(*b*) Since U is the integral of m with respect to x, what is m in terms of U? If a graph were drawn showing U as a function of x, what would represent m?

2. Find U in (86) if $m=k(a-x)$. Also if $m=k(a-x)^4$. Also if $m=k(2^{a-x}-1)$.

In Ex. 3–9, express each desired quantity as an integral.

3. The intensity i of an emotion, — *i.e.*, the amount experienced per unit time, — varies from moment to moment. Express the total amount E in T hr.

4. The price (\$$p$ per lb.) which a man would pay for a small amount of a commodity varies with the quantity (x lb.) already acquired. Starting with none, how much would he pay for X lb.?

5. The rate (l hr. per additional bushel) at which labor must be expended to produce a slight increase in a farm crop varies with the size of the crop (x bu.). Express the total labor required for a crop of any size (C bu.).

6. Express the total crop produced by any amount of labor (L hr.), if the rate of productivity (p bu./hr.) for a small increase in labor varies with the amount (x hr.) already applied.

7. Show that the accumulated value of the man's net contribution in (A) of § 86 with 5% compound interest to the time of his death would be

$$\int_0^L (R_p - R_c)(1.05)^{L-t}\,dt.$$

What would it be at any interest rate r, if he died at age 70?

8. A man considers buying a piano now. Let P be the probability of its giving him t yr. hence, a degree of happiness valued at some rate (\$$H$ per yr.) which varies with t. Discounting on a 6% basis, what is the probable present value of the piano to him for the next 15 yr.?

9. Like Ex. 8 if planning a house, to be finished $\frac{1}{2}$ yr. hence, and discounting at $5\frac{1}{2}$%.

10. In (D), p. 146, what is the probability that Y occurred during the period A.D. 206–210? A.D. 221–225? Why should the latter probability be the larger?

11. In a telephone exchange during a "busy hour" ($t=0$ to $t=60$), calls come in uniformly at very short intervals. If a certain subscriber calls twice, at random times during the hour, what is the probability that his second call will be between $t=10$ and $t=20$? Between $t=40$ and $t=50$?

12. (a) Find the amount accumulated after 30 yr. on money deposited with great frequency at the rate of \$300 per yr., if interest is at 5%. (b) The same for 20 yr. at 6%.

13. A large number of persons (N) are expected to die during a year, at equal tiny intervals. The estate of each is to be paid immediately \$1000 t, where t is the fraction of the year already elapsed. On a 5% basis what is the total present value at the beginning of the year, for all the expected payments?

§ 87. Remarks on Chapter III. We define an integral as the limit of a sum, of a certain type. But in any ordinary case we get its value by reversing a differentiation, — *i.e.*, by finding a quantity whose derivative is the integrand.

For convenience in setting up an integral, we usually consider the required quantity as the sum of many tiny elements, in each of which some variable is regarded as momentarily constant. In reality the required quantity is the *limit* of a corresponding sum. But we get the exact value by integrating instead of actually summing.

The applications of integration in science are even more varied than the illustrations which we have considered would suggest. Indeed, the procedure which we have followed is so common as to be virtually *a mode of thought.*

Of the various integrals which we have set up, some are more basic than others. The principles used in §§ 69, 71, 73, 77, 80–81, 83 should be firmly fixed in mind. Likewise we should occasionally review the memory list of integration formulas, §§ 56, 62–64. No one should waste time going to a table for integrals which ought to be written at sight.

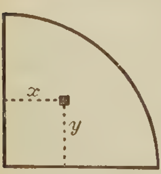


FIG. 58.

Many physical quantities are not expressible as integrals by considering strip elements such as we have used. Thus, in finding the mass of a plate whose surface density D varies with both x and y, if we choose a strip either horizontally or vertically, its tiny mass will usually be unknown. We must analyze such a plate into still smaller elements (Fig. 58), and integrate twice.

This will be taken up in Chapter V; but we need first to extend our knowledge of *methods of integration*, as in Chapter IV.

EXERCISES FOR REVIEW

1. Test the consistency of each following pair of functions as possible integrals of one quantity:

(a) $\log \dfrac{\sqrt{3x^2+x+5}+x\sqrt{3}-\sqrt{5}}{\sqrt{3x^2+x+5}+x\sqrt{3}+\sqrt{5}}$, $\log \dfrac{\sqrt{3x^2+x+5}-x\sqrt{3}-\sqrt{5}}{\sqrt{3x^2+x+5}-x\sqrt{3}+\sqrt{5}}$;

(b) $2\tan^{-1}x+\sin^2 x+1$, $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)-\cos^2 x$.

2. Do formulas (6) and (7), § 57, and (65)–(69), § 81, meet the homogeneity test of § 6? Does the equation, $dS=2\pi R\,ds$? (§ 83.) What "dimensions" has an area?

3. The linear density of a rod 40 cm. long, x cm. from one end A, is $D=6+.012\,x$. Find its centroid, and its moment of inertia about a perpendicular axis through A.

4. The vertical face of a dam has the shape bounded by $y=.01\,x^2$ and $y=16$. (The unit is 1 ft.) When water reaches the top, find the total force against the dam; also the center of pressure.

5. An iron casting with a hole through the middle has the shape generated by revolving about a line 4 units to the left of the Y-axis the part of the area bounded by $y=x^2$ and $y=4$, which lies in the first quadrant. (The unit is 1 cm.) Find

(a) The volume; (b) $\bar{y}$ for the centroid of the casting;
(c) The area of the curved outer surface.

6. A beam 300 in. long rests upon piers at its ends. The loading is 40 lb. per in. Find the bending moment 100 in. from either end.

7. Write an expression for each of the following

(a) The total amount of pain experienced in T hr. from an injury if the intensity gradually decreases.

(b) The total energy used in T hr. if the power varies. [Power is energy used, or work done, per unit of time.]

8. Find the following integrals, using tables if helpful:

(a) $\displaystyle\int \cos^{12}\theta\sin\theta\,d\theta$, (b) $\displaystyle\int \cos^4\theta\,d\theta$, (c) $\displaystyle\int \sin 6\,\theta\cos 2\,\theta\,d\theta$,

(d) $\displaystyle\int (x^2+1)^{\frac{7}{2}}\,x\,dx$, (e) $\displaystyle\int (x^2+1)^{\frac{5}{2}}\,dx$, (f) $\displaystyle\int \frac{(x+5)dx}{\sqrt{x^2+16}}$.

9. Reduce or transform each of the following, and integrate:

$e^{2x}(e^{3x}+1)dx$, $\dfrac{e^{3x}}{e^x+1}\,dx$, $\dfrac{e^x dx}{e^x+e^{-x}}$. [What does e^{-x} mean?]

10. Integrate: $(3\,x+11)\,dx/(x^2+4\,x+8)$.

Remark. The final partial fractions just found could easily be integrated; and thereby also the original fraction:

$$\int \frac{(10\,x+142)\,dx}{2\,x^3-7\,x^2-14\,x+40} = -9\log(x-2)+7\log(x-4)+2\log(2x+5)+C'.$$

We could condense this result by noting that $7\log(x-4)$ equals $\log(x-4)^7$, etc.; that adding logarithms gives the logarithm of a product; and that C' is the logarithm of some number k. Thus we could write finally:

$$\log \frac{k(x-4)^7(2\,x+5)^2}{(x-2)^9}.$$

§ 90. A Short Cut. The numerators A, B, C, in (1) above could have been found more quickly without multiplying out and equating like powers. Simply substitute the roots 2, 4, $-\frac{5}{2}$ for x in (2). For instance,

$$10(2)+142 = A(-2)(9)+B(0)+C(0),$$

or $162=-18\,A$, whence $A=-9$, as found above. And so on.

This substitution process, however, is not effective with some types of denominators; while the former process always applies.

Whichever method we use, it is well to check the final partial fractions, at least by substituting one more value for x, both in them and in the original fraction.

EXERCISES

1. Find each of the following integrals:

(*a*) $\displaystyle\int \frac{(13\,x+41)\,dx}{(x-3)(x+1)(x+2)}$, (*b*) $\displaystyle\int \frac{(14\,x-19)\,dx}{(x-2)(x-1)(x+4)}$,

(*c*) $\displaystyle\int \frac{2\,dx}{x^3+2\,x^2-3\,x}$, (*d*) $\displaystyle\int \frac{(3\,x^2+7\,x)\,dx}{x^3+6\,x^2+11\,x+6}$,

(*e*) $\displaystyle\int \frac{(3\,x^2-13\,x)\,dx}{x^3-19\,x+30}$, (*f*) $\displaystyle\int \frac{(x^2+2)\,dx}{2\,x^3-9\,x^2+3\,x+4}$,

(*g*) $\displaystyle\int \frac{(2\,x^3+1)\,dx}{x^3-9\,x}$, (*h*) $\displaystyle\int \frac{(1-8\,x^4)\,dx}{x-4\,x^3}$.

2. Evaluate the definite integrals:

(*a*) $\displaystyle\int_0^1 \frac{(11\,x^2+17\,x)dx}{(5+2\,x)(8-x)(2+x)}$, (*b*) $\displaystyle\int_1^2 \frac{(7\,x^2-3)dx}{3\,x+8\,x^2-3\,x^3}$,

(*c*) $\displaystyle\int_4^5 \frac{252\,dx}{x^4-15\,x^2+10\,x+24}$, (*d*) $\displaystyle\int_3^4 \frac{x^2dx}{25-x^2}$.

3. Find each of the following integrals in two different ways:

(*a*) $\displaystyle\int \frac{(2\,x+9)dx}{x^2-5\,x+6}$, (*b*) $\displaystyle\int \frac{(5\,x+11)dx}{x^2-3\,x-10}$.

4. The time required to form a quantity X of a certain chemical is given by the integral:

$$t=k\int_0^X \frac{dx}{(a-x)(b-x)},$$

where a, b, and k are constants. Take $a=3$, $b=2$, and work out the integration.

5. In Ex. 4 take $a=.8$, $b=.7$, and integrate.

6. Obtain the general solution in Ex. 4 in terms of a and b.

§ 91. Quadratic Factors. If we had to separate

$$\frac{2\,x-58}{(x-3)(x^2+4\,x+5)}$$

into partial fractions, it would be *possible* to factor $(x^2+4\,x+5)$ as $(x+2+\sqrt{-1})(x+2-\sqrt{-1})$. But this would be inadvisable.

For such a factorization would give finally

$$\frac{2\,x-58}{(x-3)(x^2+4\,x+5)}=-\frac{2}{x-3}+\frac{1+6\sqrt{-1}}{x+2+\sqrt{-1}}+\frac{1-6\sqrt{-1}}{x+2-\sqrt{-1}}, \tag{3}$$

the correctness of which can be verified by addition. And the last two fractions would lead to imaginary logarithms when integrated.

A better plan is to keep the quadratic $(x^2+4\,x+5)$, or any similar quadratic, unfactored, as the denominator of a single fraction. In effect that fraction will be the real sum of two imaginary linear fractions, like the last two in (3). Hence its numerator will usually contain x and have the form $Ax+B$.

In fact, the last two fractions in (3) could be replaced by

$$\frac{2x+16}{x^2+4x+5}, \tag{4}$$

a single real fraction of the form mentioned. (Verify.)

In general, then, for any quadratic factor of a given denominator, whose roots are imaginary, we assume a single partial fraction with that quadratic as the denominator, and with a linear expression like $Ax+B$ for the numerator. The work thereafter runs as in § 89, with the use of the short cut of § 90 for any linear fraction present.

We can, if we like, proceed similarly when a quadratic factor has roots which are real but irrational. Some labor with radicals is thus avoided, but no such obstacle as imaginary logarithms.

Ex. I. Separate $\dfrac{x^2+11x+6}{(x-4)(x^2+4x+1)}$.

We assume

$$\frac{x^2+11x+6}{(x-4)(x^2+4x+1)} = \frac{A}{x-4} + \frac{Bx+C}{(x^2+4x+1)}. \tag{5}$$

$$\therefore \qquad x^2+11x+6 = A(x^2+4x+1)+Bx(x-4)+C(x-4).$$

Putting $x=4$, we find $66=33A$; or $A=2$. Equating x^2 terms on both sides, and likewise the constants:

$$1 = A+B, \qquad \therefore\ B=-1,$$
$$6 = A-4C, \qquad \therefore\ C=-1.$$

$$\therefore\ \frac{x^2+11x+6}{(x-4)(x^2+4x+1)} = \frac{2}{x-4} - \frac{x+1}{x^2+4x+1}. \qquad (\textit{Check?})$$

Remark. To integrate the latter fraction we would split it:

$$-\frac{1}{2}\frac{(2x+4)}{x^2+4x+1} + \frac{1}{x^2+4x+1}. \qquad (\text{Cf. § 65.})$$

The first of these fractions would give $-\frac{1}{2}\log(x^2+4x+1)$. The second, on completing the square, would give

$$\frac{1}{2\sqrt{3}}\log\frac{x+2-\sqrt{3}}{x+2+\sqrt{3}}. \tag{6}$$

EXERCISES

1. Find the following integrals:

(a) $\int \frac{x^4dx}{x^3-8}$, (b) $\int \frac{x^4dx}{x^4-16}$, (c) $\int_0^1 \frac{x^3dx}{x^2+4}$.

2. Integrate the following fractions with respect to x:

(a) $\frac{7x+41}{(x+2)(x^2+4x+13)}$, (b) $\frac{x^2-40x-7}{(x^2+9)(x^2+1)}$,

(c) $\frac{12x^2}{(x^2-4)(x^2+2)}$, (d) $\frac{2x^2+4}{x^4+6x^2+5}$,

(e) $\frac{3x^3-5x^2-46x+63}{x^4-21x^2+44x-24}$, (f) $\frac{x^3+5x-2}{6+x^2-x^4}$,

(g) $\frac{3x+7}{x^4-2x^2-9x+10}$, (h) $\frac{x^2+9}{x^4-4x^3-4x-1}$.

3. Find the following integrals:

(a) $\int \frac{dx}{x^4+16}$, (b) $\int \frac{x^2\,dx}{x^4+5x^2+9}$.

[Hint: In each case express the denominator as the difference of two squares by adding a suitable middle term and later subtracting it.]

4. Separate and integrate: $\frac{x\,dx}{x^3-5}$.

[Suggestion: To avoid much work with radicals, replace the number 5 by k^3 until through integrating. Then replace each k by $\sqrt[3]{5}$.]

5. Like Ex. 4 for each of the following:

(a) $\frac{dt}{t^3-16}$, (b) $\frac{dx}{x^3+4}$, (c) $\frac{y^2\,dy}{y^4-7}$.

6. For review, integrate the following by inspection:

(a) $\frac{x\,dx}{\sqrt{x^2+16}}$, (b) $\frac{x\,dx}{\sqrt{x^4+16}}$, (c) $\frac{dx}{\sqrt{9-2x^2}}$.

7. Integrate the quantity: $\frac{e^x\,dx}{(e^x+1)(e^{2x}+4)}$.

[Hint: Denote e^x temporarily by u, and hence the numerator by du.]

8. A flat plate has the shape of the area under the curve $y=\frac{80}{x^2+4}$ from $x=0$ to $x=4$. Its surface density varies thus: $D=\frac{20}{x+12}$. Find its moment of inertia about the Y-axis.

9. A plate has the shape of the area between the curves $y=\pm\frac{20}{x+1}$ from $x=0$ to $x=4$. Its surface density varies thus: $D=\frac{8}{x+4}$. Find its centroid.

10. Separate $\frac{2x-58}{(x-3)(x^2+4x+5)}$ into partial fractions and compare (3), (4), p. 153.

§ 92. Repeated Factors. Consider the fraction

$$\frac{3x^2+11}{(x+1)(x-2)^2}, \tag{7}$$

whose denominator has a repeated linear factor $(x-2)^2$.

Clearly this might arise from adding the fractions

$$\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}. \tag{8}$$

For, the lowest common denominator would be simply $(x+1)(x-2)^2$, and the combined numerator some second-degree quantity.

It would not do to use as the third fraction simply $C/(x-2)$; for then the least common denominator would be merely $(x+1)(x-2)$.

Conceivably the second fraction in (8) might be omitted; but it may be needed in order to produce the exact *numerator* in (7).

To be safe when separating a fraction like (7) we assume the complete form (8), which covers all possible needs. Whenever the second fraction happens to be superfluous, B will come out zero in solving, and the superfluous fraction will disappear automatically.

Though the third fraction in (8) has a denominator of the second degree, a constant numerator suffices. For, if we assumed $Cx+D$, we could reduce.

Thus $$\frac{Cx+D}{(x-2)^2}=\frac{C(x-2)+(2C+D)}{(x-2)^2}=\frac{C}{x-2}+\frac{2C+D}{(x-2)^2}.$$

Then $\frac{C}{x-2}$ would combine with $\frac{B}{x-2}$ in (8) to make merely one fraction of that type; while $(2C+D)$ is merely a constant, as assumed in (8).

For *any* repeated denominator we can always use the same type of numerator as if the denominator were not repeated. And when there is a repeated linear factor $(x-k)^n$, we assume all the fractions,

$$\frac{A_1}{x-k}+\frac{A_2}{(x-k)^2}+\cdots+\frac{A_n}{(x-k)^n}.$$

For similar reasons, when there is a repeated quadratic factor $(x^2+px+q)^m$, whose linear factors would be troublesome, we assume

$$\frac{A_1x+B_1}{x^2+px+q}+\frac{A_2x+B_2}{(x^2+px+q)^2}+\cdots+\frac{A_mx+B_m}{(x^2+px+q)^m}. \tag{9}$$

§ 93. Character of the Integrals. The partial fractions into which more complicated fractions are separated are of only four essentially different types:

$$\underset{(I)}{\frac{A}{x-k}}, \qquad \underset{(II)}{\frac{B}{(x-k)^n}}, \qquad \underset{(III)}{\frac{Cx+D}{x^2+px+q}}, \qquad \underset{(IV)}{\frac{Ex+F}{(x^2+px+q)^n}}, \quad n>1.$$

All of these can be integrated: (*I*) gives a logarithm; (*II*) gives another negative power, not a logarithm; (*III*) and (*IV*) each give two integrals, by splitting their numerators into two parts, one of which supplies the differential of the denominator quantity x^2+px+q, and the other of which is some constant. The first part of (*III*) then gives a logarithm; of (*IV*), another negative power. The second part of (*III*) gives an arctangent or a logarithm, according as x^2+px+q is the sum or difference of two squares. The second part of (*IV*) is handled by a Reduction Formula or by another method to be shown later, and gives fractional forms plus a final integral like that from the second part of (*III*).

Instead of x^2+px+q the tables print ax^2+bx+c to avoid the necessity of factoring out the coefficient of x^2 when there is one present, other than unity.

It can be proved that for every rational fraction whose numerator is of lower degree than the denominator, there exist partial fractions of the types above. Hence every such fraction is theoretically integrable. But practically we have the algebraic impossibility of factoring the given denominator in some complicated cases, and must then resort to approximations.

§ 94. **Fractions Not Requiring Separation.** Some fractions are type forms in disguise, and can be integrated quickly without forming partial fractions.

Ex. I. $$\int \frac{x^2\,dx}{x^6-64}.$$

Taking x^3 as u, this is simply

$$\frac{1}{3}\int \frac{du}{u^2-64} = \frac{1}{48}\log\frac{u-8}{u+8}+C.$$

To integrate by means of standard partial fractions would be tedious.

Ex. II. $$\int \frac{4\,dx}{(x^2+5)^3}.$$

This is already one of the *basic* partial fractions, and should be handled as such. To attempt to break it up would be a sheer waste of time.

EXERCISES

1. Integrate with respect to x the following fractions:

(*a*) $\dfrac{49}{(x+3)^2(x-4)}$, (*b*) $\dfrac{x^3+2x}{(x-1)^2(x+2)}$,

(*c*) $\dfrac{x^3+9}{x^2(x-3)^2}$, (*d*) $\dfrac{4x}{(x+1)(x^2-1)}$,

(*e*) $\dfrac{x^2}{(x+2)^3}$, (*f*) $\dfrac{3x+12}{(x-2)^2(x^2+5)}$,

(*g*) $\dfrac{6x-5}{x^4+4x^3+5x^2}$, (*h*) $\dfrac{2(x^2+1)}{(3x^2+5)^2}$,

(*i*) $\dfrac{2x^3}{x^4+8x^2+16}$, (*j*) $\dfrac{x^3+29x+32}{x^4+x^2+36x+52}$,

(*k*) $\dfrac{6}{(x-2)^3}$, (*l*) $\dfrac{256}{x^4-22x^2+24x+45}$,

(*m*) $\dfrac{4x^3+7x^2+5}{x^4+2x^2+1}$, (*n*) $\dfrac{6x^2+9}{x(x^2+3)^2}$, (*o*) $\dfrac{8}{(x^2+2)^3}$.

2. Evaluate the following integrals:

(*a*) $\displaystyle\int_3^4 \frac{x^2\,dx}{x^3-8}$, (*b*) $\displaystyle\int_4^5 \frac{x\,dx}{x^2-9}$, (*c*) $\displaystyle\int_0^2 \frac{x^4\,dx}{x^5+32}$,

(*d*) $\displaystyle\int_0^1 \frac{x^4\,dx}{x^{10}+16}$, (*e*) $\displaystyle\int_{2a}^{4a} \frac{x^2\,dx}{x^6-a^6}$, (*f*) $\displaystyle\int_{2a}^{3a} \frac{x\,dx}{x^4-a^4}$.

3. Find $\bar{x}$ for the centroid of the area under the curve $y=\dfrac{400}{(x+5)^2}$ from $x=0$ to $x=5$.

4. Transform each of the following expressions into a form readily integrable on making some simple substitution:

(*a*) $\dfrac{dx}{e^x+e^{-x}}$, (*b*) $\dfrac{dx}{e^{2x}+1}$, (*c*) $\dfrac{e^{3x}\,dx}{e^x+1}$.

5. Separate the fraction $F=\dfrac{(x^2-7)(x-3)(x-5)}{(x-4)^{10}}$ into partial fractions by temporarily putting $x-4=t$, or $x=t+4$ and reducing.

6. Like Ex. 5 for the fraction $\dfrac{x^3+6x+1}{(x-3)^4}$. Compare the resulting numerators with the result of dividing x^3+6x+1 by $x-3$ synthetically four times, as in Horner's method. (*Intro.*, §§ 239, 240.)

7. Integrate $\dfrac{x^2\,dx}{(x^2+1)^3}$. [Use a reduction formula at once.]

8. Show the *form* of the partial fractions which you would use if separating

(*a*) $\dfrac{x^4}{(x-2)^2(x^2+1)^3}$, (*b*) $\dfrac{x^7+4}{x^6-64}$, (*c*) $\dfrac{x^7+4}{x^2(x^2+3x)(x^2+2x-3)(x^2+9)}$.

9. Like Ex. 8 for $x/(x^5-1)$. [Hint: Find the roots of $x^5-1=0$ by *Intro.*, § 352. Then x minus each root is a linear factor of the denominator. But corresponding complex factors, such as $(x-\text{cis}\,72°)$ and $(x-\text{cis}\,288°)$, for instance, have real quadratic products.]

PART II. IRRATIONAL ALGEBRAIC FORMS

§ 95. Linear Radicals. Most algebraic expressions which involve even a simple radical with some function of x outside are not integrable by tables alone. The general procedure

in such cases is to *make a substitution*, if possible, which will reduce the entire integrand to a *rational* function of some new variable.

A very common substitution is to let the given radical equal a new variable t. This works in all cases in which the quantity inside the given radical is linear or a linear fraction; also in some other cases.

Ex. I. $$F=\int\frac{\sqrt{x+3}}{x+4}\,dx. \qquad (10)$$

Let $\sqrt{x+3}=t$; *i.e.*, $x=t^2-3$.

Then $x+4=t^2+1$, and $dx=2\,t\,dt$.

Substituting these several values will reduce the integral to

$$F=\int\frac{2\,t^2\,dt}{t^2+1}=2\,t\ -2\tan^{-1}t\ +C. \qquad (11)$$

I.e., $$F=2\sqrt{x+3}\ -2\ \tan^{-1}\sqrt{x+3}\ +C.$$

N.B. *It is necessary to substitute for dx.* For, the original notation indicates that the derivative of F with respect to x is $\sqrt{x+3}/(x+4)$; and of course the derivative with respect to t will not be the mere equivalent of this, but rather that equivalent multiplied by dx/dt or $2\,t$.

§ 96. **Change of Limits.** In finding a definite integral by a substitution, we need not change back to the original variable x after integrating. Merely substitute for the new variable t its values which correspond to the original limits for x.

E.g., suppose we desired the integral in (10) above between the limits 1 and 6. Having put $\sqrt{x+3}=t$, we see that

when $x=1$, $t=2$; when $x=6$, $t=3$.

Thus we could substitute 2 and 3 for t in (11) without changing back to the radical involving x.

In complicated cases this device often saves much time.

EXERCISES

Tables may be used wherever helpful.

1. Find the following integrals:

(*a*) $\displaystyle\int \frac{dx}{(x+2)\sqrt{x-2}}$, (*b*) $\displaystyle\int_0^1 \frac{(x^2+5\,x)dx}{\sqrt{x+4}}$,

(c) $\displaystyle\int \frac{\sqrt{x-7}}{x^2-16}\,dx$, (*d*) $\displaystyle\int_4^{20} \frac{dx}{(x^2-4)\sqrt{x+5}}$,

(*e*) $\displaystyle\int \frac{2+x}{\sqrt[5]{1-x}}\,dx$, (*f*) $\displaystyle\int_0^{13} \sqrt[3]{2\,x+1}\,x\,dx$,

(*g*) $\displaystyle\int \frac{\sqrt{x}+2\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx$, (*h*) $\displaystyle\int_0^4 \frac{(1+\sqrt{x})^2}{x+9}\,dx$,

(*i*) $\displaystyle\int \sqrt{1+\sqrt[3]{x}}\,dx$, (*j*) $\displaystyle\int x\sqrt{3+\sqrt{x+1}}\,dx$,

(*k*) $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{x}}}$, (*l*) $\displaystyle\int \frac{\sqrt{5+\sqrt{x+4}}}{x}\,dx$.

2. Derive formulas (6) and (9), p. 492, in the special cases:

(*a*) $\displaystyle\int \frac{x\,dx}{(7+4\,x)^{\frac{3}{2}}}$, (*b*) $\displaystyle\int \sqrt{\frac{x+5}{x+1}}\,dx$.

3. Find the integral in Ex. 2(*b*) also by first rationalizing the numerator algebraically, and then splitting as in § 65.

4. Find the integral of $\dfrac{\sqrt{x^4+1}}{x}\,dx$.

[Hint: Let the radical equal t and show that the entire integrand reduces to $\dfrac{t^2\,dt}{2(t^2-1)}$.]

5. Find the integrals:

(*a*) $\displaystyle\int \frac{\sqrt{e^x+1}}{e^x-3}\,e^x\,dx$, (*b*) $\displaystyle\int \frac{\sqrt{e^x+9}}{e^x}\,dx$.

6. A flat plate has the shape bounded by the parabola $y^2=9\,x$ and the line $x=4$. Its surface density varies thus: $D=5/(x+1)$. Find

(*a*) The moment of inertia about the Y-axis;
(*b*) The mass; (*c*) $\bar{x}$ for the centroid.

7. In § 96 why must $t=2$ and not -2, when $x=1$? [What does the symbol $\sqrt{4}$ denote?]

§ 97. Higher Radicals. The substitution of t for a radical will sometimes rationalize an expression involving a non-linear radical quantity. The only important case is that in which the radical quantity is a binomial of the form ax^n+b, and the outside function is a single power x^m, suitably related to the lone power x^n inside. Such an integrand may be written

$$x^m(ax^n+b)^p dx, \tag{12}$$

where p is any fractional exponent, say r/s.

In case $m=n-1$, the outside factor furnishes the differential of ax^n+b, aside from a constant multiplier, and we have a simple type form $u^p du$.

If m differs from $n-1$ by any multiple of n, the substitution

$$(ax^n+b)^{\frac{1}{s}}=t \tag{13}$$

will be effective. This will be clear from Ex. I below.

There is another possibility: Factor out the inside power x^n, or really $(x^n)^p$. The binomial will then be $a+b/x^n$, or $a+bx^{-n}$, making the integrand read:

$$x^{m+np}(a+bx^{-n})^p dx. \tag{14}$$

Then apply the above ideas to this new form as if it had been the original integrand. (See Ex. II below.)

Remark. Any form which could be rationalized after using a reduction formula can also be rationalized before. But sometimes a less complicated rational form can be secured by first using a reduction formula to cut down any high powers.

Ex. I. $$F=\int\frac{(2\,x^4-27)^{\frac{5}{3}}\,dx}{x}.$$

The differential of $2\,x^4-27$ is $8\,x^3\,dx$. The outside power x^{-1} has an exponent differing from 3 by 4, a "multiple" of the inside exponent 4.

First rewrite F, supplying the differential $8\,x^3\,dx$:

$$F=\frac{1}{8}\int\frac{(2\,x^4-27)^{\frac{5}{3}}8\,x^3\,dx}{x^4}.$$

Now put $\qquad (2\,x^4-27)^{\frac{1}{3}}=t, \qquad (15)$

i.e., $\qquad 2\,x^4=t^3+27. \qquad (16)$

$\therefore \qquad 8\,x^3\,dx=3\,t^2\,dt. \qquad (17)$

Then F becomes

$$F=\frac{1}{8}\int\frac{t^5(3\,t^2\,dt)}{\frac{1}{2}(t^3+27)}=\frac{3}{4}\int\frac{t^7\,dt}{t^3+27}.$$

By dividing out and using partial fractions we could integrate.

Ex. II. $$F=\int\frac{(1+x^2)^{\frac{5}{2}}\,dx}{x^{10}}.$$

The outside exponent -10 differs from the desired differential exponent $+1$ by 11, which is not a multiple of 2.

Factor out $(x^2)^{\frac{5}{2}}$, and the integrand becomes

$$\frac{(x^{-2}+1)^{\frac{5}{2}}\,x^5\,dx}{x^{10}};$$

or $$(x^{-2}+1)^{\frac{5}{2}}\,x^{-5}\,dx.$$

Since the differential of $x^{-2}+1$ is $-2\,x^{-3}\,dx$, write:

$$F=-\tfrac{1}{2}\int(x^{-2}+1)^{\frac{5}{2}}\,x^{-2}(-2\,x^{-3}\,dx). \qquad (18)$$

Now let $(x^{-2}+1)^{\frac{1}{2}}=t$; that is, $x^{-2}=t^2-1$, whence $-2\,x^{-3}\,dx=2\,t\,dt$.

$$\therefore F=-\tfrac{1}{2}\int t^5(t^2-1)(2\,t\,dt)=-\int(t^8-t^6)\,dt;$$

i.e., $$F=-\tfrac{1}{9}\,t^9+\tfrac{1}{7}\,t^7+C,\ =\frac{t^7}{63}(9-7\,t^2)+C.$$

To return to x, we recall that

$$t=\left(\frac{1}{x^2}+1\right)^{\frac{1}{2}};\ \text{or, simplified, } t=\frac{\sqrt{1+x^2}}{x}. \qquad (19)$$

EXERCISES

1. Find the integral $\int x^8\sqrt{x^3+1}\,dx$ by rationalizing directly; also by first using a reduction formula.

2. Integrate each of the following, using tables wherever helpful:

(*a*) $\dfrac{x^7\,dx}{\sqrt{x^4+3}}$, (*b*) $\dfrac{\sqrt{x^4-9}}{x}\,dx$, (*c*) $\dfrac{dx}{x\sqrt{x^6+1}}$,

(*d*) $\dfrac{\sqrt{3-2x^2}}{x^4}\,dx$, (*e*) $\dfrac{x^4\,dx}{(x^2+1)^{\frac{7}{2}}}$, (*f*) $\dfrac{(x^2-1)^{\frac{3}{2}}}{x^6}\,dx$,

(*g*) $\dfrac{(x^4+1)^{\frac{5}{2}}\,dx}{x^{15}}$, (*h*) $\dfrac{(x^2+4)^{\frac{5}{2}}}{x^6}\,dx$, (*i*) $\dfrac{dx}{(x^2+4)^{\frac{5}{2}}}$.

3. Integrate by inspection:

$$\frac{dx}{x\sqrt{x^2-25}},\quad \frac{t^2\,dt}{\sqrt{16-t^6}},\quad \frac{y\,dy}{\sqrt{y^4-9}},\quad \frac{u\,du}{\sqrt{5-3u^2}},\quad \frac{e^x\,dx}{\sqrt{4e^x-e^{2x}}}.$$

4. Writing the integral $\int\sqrt{x^2-4}\,dx$ in the form

$$\int(x+2)\sqrt{\frac{x-2}{x+2}}\,dx,$$

can you see any substitution which would rationalize the integrand? What equivalent substitution could be made for the original radical?

5. A flat plate has the shape bounded by the curve $y^2=x^3-8$ and the line $x=3$. Its surface density varies inversely as x, and is 2 when $x=3$. Find the mass of the plate.

§ 98. The Radical $\sqrt{x^2+px+q}$. Most integrands involving a quadratic radical cannot be rationalized by setting the radical equal to t. But they can all be rationalized by some other substitution, — it being understood, of course, that no other radical is present.

Consider first the case where the x^2 term under the radical is *positive*. And let its coefficient be reduced to $+1$, by factoring out some number if necessary. We then have the radical

$$\sqrt{x^2+px+q}.$$

The best procedure in this case is to *let the radical equal* $(t-x)$. This will make both x and dx rational functions of t, as will be clear from the following example.

Ex. I.
$$F=\int\frac{dx}{(x+1)\sqrt{x^2+x+1}}. \tag{20}$$

Let $\sqrt{x^2+x+1}=t-x$; or $x^2+x+1=t^2-2tx+x^2$.

Simplifying and solving for x we find:

$$x=\frac{t^2-1}{2\,t+1}. \tag{21}$$

Thus we are in reality substituting this quantity for x; and from it we obtain values in terms of t for all the quantities which appear in (20).

$$\therefore \qquad dx=\frac{2(t^2+t+1)dt}{(2\,t+1)^2}. \tag{22}$$

Also (21) gives for $(x+1)$, and for the radical or $(t-x)$:

$$x+1=\frac{t^2+2\,t}{2\,t+1}, \qquad t-x=\frac{t^2+t+1}{2\,t+1}. \tag{23}$$

Using the values (21), (22), (23) in (20), and simplifying:

$$F=2\int\frac{dt}{t^2+2\,t}=2\int\frac{dt}{(t+1)^2-1}=\log\frac{t}{t+2}+C.$$

By the original substitution, we now put $t=\sqrt{x^2+x+1}+x$.

§ 99. The Radical $\sqrt{-x^2+px+q}$. If in a quadratic radical quantity the x^2 term is negative, it will be useless to let the radical equal $t-x$. (Why?)

If such a radical is real at all, the quadratic must be positive for some values of x. And as it is negative for very large values of x, due to the $-x^2$, it must become zero somewhere between. Thus the quadratic must have real roots, and hence also real linear factors, say

$$(x-a)(b-x).$$

To rationalize any integrand involving the radical in question, and no other radical, we may *let the radical equal either of these factors times a new variable t.*

For the radical could be written thus:

$$\sqrt{-x^2+px+q}=\sqrt{(x-a)(b-x)}=\sqrt{\frac{x-a}{b-x}}(b-x). \tag{24}$$

And we know we could rationalize by letting this latter radical equal t, since the quantity inside is merely a linear fraction. (§ 95.) That is, we may let the original radical equal $(b-x)t$, as stated. Similarly if we prefer to use $(x-a)t$.

This plan can also be used for a quadratic radical having a positive x^2 term, if the linear factors are real.

Many integrals involving a quadratic radical with a single outside power of x can be obtained from tables directly or by using Reduction Formulas (44)–(49), p. 495. If the outside function is the sum of several terms, we can break it up and deal with its terms individually.

Ex. I. $$F=\int\frac{dx}{(x+3)\sqrt{4-x^2}}.$$

Let $\sqrt{4-x^2}=(2+x)t$. Squaring and solving for x:

$$x=\frac{2-2\,t^2}{1+t^2}. \qquad \therefore\ dx=-\frac{8\,t\,dt}{(1+t^2)^2}. \tag{25}$$

Also $$2+x=\frac{4}{1+t^2}. \qquad \therefore\ \sqrt{4-x^2}=\frac{4\,t}{1+t^2}.$$

Similarly we find $x+3=(5+t^2)/(1+t^2)$. Finally, then, substituting for dx, $x+3$, and the radical:

$$F=-\int\frac{8\,t\,dt}{(1+t^2)^2}\cdot\frac{1+t^2}{5+t^2}\cdot\frac{1+t^2}{4\,t}=-2\int\frac{dt}{5+t^2}.$$

I.e., $$F=-\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right)+C, \tag{26}$$

or, $$F=-\frac{2}{\sqrt{5}}\tan^{-1}\sqrt{\frac{2-x}{5(2+x)}}+C. \tag{27}$$

EXERCISES

1. Find each following integral by one of the rationalizing substitutions of §§ 98–99. Check (*a*), (*c*) by tables.

(*a*) $\displaystyle\int\frac{dx}{x\sqrt{x^2+2\,x+3}}$, (*b*) $\displaystyle\int\frac{\sqrt{x^2+x+4}}{x}\,dx$,

(*c*) $\displaystyle\int\frac{dx}{x\sqrt{4-x^2}}$, (*d*) $\displaystyle\int\frac{\sqrt{6\,x-x^2}}{x^2}\,dx$.

2. Integrate the following quantities:

(*a*) $\displaystyle\frac{dx}{(x+2)\sqrt{x^2+x+3}}$, (*b*) $\displaystyle\frac{\sqrt{x^2+x-4}}{x}\,dx$,

(*c*) $\displaystyle\frac{dx}{(x-1)\sqrt{9-x^2}}$, (*d*) $\displaystyle\frac{dx}{(x+1)\sqrt{5\,x-x^2}}$.

3. Integrate $\dfrac{dx}{(x+1)\sqrt{x^2-3x+2}}$ in two ways:

(*a*) By using $t-x$, (*b*) By using a factor times t.

4. (*a*), (*b*). Like Ex. 3 (*a*), (*b*) for the integral $\displaystyle\int \frac{\sqrt{x^2-10x+9}}{x^2}\,dx$.

5. In Ex. 1 (*a*) rationalize and integrate by letting the radical equal $t+x$. Reconcile the result with that obtained by using $t-x$.

6. (*a*), (*b*). Verify integration formulas (15), (22), p. 493, by making the rationalizing substitutions.

7. (*a*), (*b*). Verify formulas (40), (42), p. 495, by differentiating the right-hand members.

8. A flat plate has the shape bounded by the two branches of the hyperbola $4y^2-x^2=16$, the Y-axis and the line $x=3$. Its surface density varies: $D=5/(x+1)$. Find (*a*) its mass, (*b*) its moment of inertia with respect to the Y-axis, (*c*) the radius of gyration about the Y-axis, and (*d*) the position of its centroid.

9. (*a*)–(*d*). Like Ex. 8 (*a*)–(*d*) for a similar plate bounded by the ellipse $9x^2+25y^2=225$, the Y-axis, and the line $x=3$.

§ 100. More about the Tables. Integrals involving a trinomial quadratic, $Q=ax^2+bx+c$, are given on pages 494–95; also reduction formulas for products of the type

$$x^mQ^n\,dx.$$

Both m and n may have any values, $+$ or $-$, except such values as make some denominator zero. For these latter values the integrals are given separately or else are reducible to tabulated forms by substituting $x = 1/t$.

Formulas (44) and (45) reduce m to 1 or 0; or raise it to -1 or 0 if negative. Then (46) and (47) apply when $m=1$ or -1; and (48)–(49) reduce any remaining power of Q. These formulas, by the way, can be used for integral values of n as well as for radical forms.

Still other integrals can be found in larger tables. But in general integrands involving a polynomial besides $Q^n\,dx$ must be transformed. A polynomial *multiplier*, as for instance in

$$\int\left(x^3+5x^2+3+\frac{11}{x}\right)\sqrt{x^2+4x+7}\,dx,$$

may be taken term by term with the radical. But when a polynomial occurs in the *denominator*, we have to rationalize by substituting $(t-x)$ or a factor times t. Where there is a choice, the former is usually simpler when Q^n occurs in the numerator, and the latter otherwise.

Some trinomial forms are best handled by completing the square and making a linear substitution $x+a=t$. *E.g.*,

$$\int\frac{x^3dx}{(x^2+6x+11)^{\frac{5}{2}}}=\int\frac{x^3dx}{[(x+3)^2+2]^{\frac{5}{2}}}=\int\frac{(t-3)^3dt}{[t^2+2]^{\frac{5}{2}}}.$$

Expanding the numerator and forming combinations t^2+2:

$$(t-3)^3=t^3-9t^2+27t-27=t(t^2+2)-9(t^2+2)+25t-9,$$

the integral breaks up into

$$\int\frac{t\,dt}{(t^2+2)^{\frac{3}{2}}}-9\int\frac{dt}{(t^2+2)^{\frac{3}{2}}}+25\int\frac{t\,dt}{(t^2+2)^{\frac{5}{2}}}-9\int\frac{dt}{(t^2+2)^{\frac{5}{2}}}.$$

The first and third are $u^n du$ types, and the other two are quickly reduced by formula (31) for *binomials*. This takes less time than to apply the trinomial reduction formulas at the outset, if we understand the process and can use it expertly.

Remark. Every integrand which is rational except for some root of a linear quantity or fraction, or a square root of a quadratic quantity, can be rationalized. But no other radical forms can be, unless a suitable power of x is present outside the radical, as in § 97. Many integrals involving radicals of higher degree can only be approximated.

EXERCISES

1. Find the following integrals by using tables:

(*a*) $\int x(x^2+5x+2)^{\frac{3}{2}}\,dx$, (*b*) $\int x^2\sqrt{x^2-x-3}\,dx$,

(*c*) $\int\frac{dx}{x\sqrt{x^2-3x-4}}$, (*d*) $\int\frac{dx}{(x^2+x+2)^{\frac{3}{2}}}$,

(*e*) $\int\frac{x^2dx}{(1+x-x^2)^3}$, (*f*) $\int\frac{dx}{x(1-x-x^2)^{\frac{3}{2}}}$,

(*g*) $\int\frac{(x^2+x-2)^{\frac{5}{2}}}{x^2}\,dx$, (*h*) $\int\frac{3x^2+2x+7}{(x^2+2x+3)^{\frac{1}{2}}}\,dx$.

2. (*a*) Integrate $\dfrac{x^3\,dx}{(x^2+4x+5)^{\frac{3}{2}}}$ by tables.

(*b*) Check by making a linear substitution and grouping terms.

3. Integrate by inspection:

(*a*) $\dfrac{x^2\,dx}{\sqrt{4x^3-x^6}}$, (*b*) $\dfrac{x^2\,dx}{\sqrt{x^6-2x^3+5}}$.

4. Integrate by tables or otherwise:

(*a*) $\dfrac{x^2\,dx}{(x^2+2x+2)^{\frac{3}{2}}}$, (*b*) $\dfrac{x^3\,dx}{(x^2+7)^3}$.

5. (*a*) Integrate $\dfrac{(x^2+5)dx}{(x^2+3x+4)^2}$. (*b*) Check by differentiating your result.

PART III. TRIGONOMETRIC FORMS

The more important trigonometric integrands fall into four classes: (1) Positive or negative integral powers of a function of one angle, and products of such powers; (2) Other rational expressions involving one angle; (3) Certain products involving two or more angles; (4) Certain fractional powers and irrational forms. These classes will be discussed separately.

§ **101. Powers of Functions.** The general method of integrating a power of any trigonometric function is to use a reduction formula. (§ 67.) Likewise for products and quotients of powers, after first changing all functions into sines and cosines.

E.g., $\dfrac{\tan^2 x \sec^3 x}{\csc^4 x}\,dx$ would be written $\dfrac{\sin^2 x}{\cos^2 x}\cdot\dfrac{1}{\cos^3 x}\cdot\dfrac{\sin^4 x}{1}\,dx$, or
$$\frac{\sin^6 x\,dx}{\cos^5 x}.$$
Formulas (66) and (69), p. 497, would ultimately reduce this to $dx/\cos x$, or $\sec x\,dx$, a type form.

Reduction formulas, however, are not as quick in certain cases as the methods indicated below, which require no tables. Study the numerical examples until you see clearly *why* each

method works and what trouble would be encountered if we had even powers instead of odd, or odd instead of even. Then it will be easy to remember these valuable quick methods permanently, — not by sheer memorizing, but by understanding.

(I) *An odd positive power of* $\sin x$, alone, or multiplied or divided by any power of $\cos x$: Split off $-\sin x\,dx$, the differential of $\cos x$; and change the remaining even power of $\sin x$ into powers of $\cos x$.

E.g., $\int \frac{\sin^5 x\,dx}{\cos^2 x}$ could be rewritten in the form

$$-\int \frac{\sin^4 x}{\cos^2 x}(-\sin x\,dx), \qquad \text{or } -\int \frac{(1-\cos^2 x)^2}{\cos^2 x}(d\cos x).$$

The integrand is now of the form $\frac{(1-u^2)^2}{u^2}\,du$, where u is $\cos x$. This can be expanded, reduced, and integrated; and u again replaced by $\cos x$.

(II) *An odd positive power of* $\tan x$, alone, or multiplied or divided by any power of $\sec x$: Split off $\tan x \sec x\,dx$, as $d(\sec x)$; and change the remaining even power of $\tan x$ into powers of $\sec x$.

$$E.g., \int \tan^7 x\,dx = \int \frac{\tan^6 x}{\sec x}(\tan x \sec x\,dx) = \int \frac{(\sec^2 x-1)^3}{\sec x}(d\sec x).$$

The integrand is now of the form $\frac{(u^2-1)^3}{u}\,du$, where u is $\sec x$.

(III) *An even positive power of* $\sec x$, alone, or multiplied or divided by any power of $\tan x$: Split off $\sec^2 x\,dx$, as $d(\tan x)$; and change the remaining even power of $\sec x$ into powers of $\tan x$.

$$E.g., \int \tan^9 x \sec^4 x\,dx = \int \tan^9 x(\tan^2 x+1)\sec^2 x\,dx = \int u^9(u^2+1)\,du.$$

Remark. Similar methods apply to the corresponding cases for the co-functions; *i.e.*, where $\sin x$, $\tan x$, and $\sec x$

are replaced above by $\cos x$, $\operatorname{ctn} x$, $\csc x$, etc. In none of these cases does the transformation introduce a radical.

With a little practice we need write only a few of the steps.

§ **102. Other Devices.** Several other combinations of *sines and cosines* can be integrated quickly without tables, when the powers involved are not very high. These more special methods need not be memorized outright, but should be practiced enough to get some idea of possible procedure.

(IV) *Even powers*, singly, or in products. Use the double-angle formulas, p. 490.

E.g., we can integrate $\sin^4 x \cos^4 x\, dx$ by writing successively:

$$\sin^4 x \cos^4 x = (\tfrac{1}{2}\sin 2x)^4 = \tfrac{1}{16}[\tfrac{1}{2}(1-\cos 4x)]^2$$
$$= \tfrac{1}{64}[1 - 2\cos 4x + \tfrac{1}{2}(1+\cos 8x)].$$

The same formulas are useful when we have to integrate a quantity involving the square root of $1-\cos\theta$ or $1-\sin\theta$. For $1-\cos\theta = 2\sin^2\dfrac{\theta}{2}$, and the radical disappears. And $1-\sin\theta$ can first be written $1-\cos u$, where $u = \dfrac{\pi}{2}-\theta$.

(V) *An even power of* $\sin x$, *divided by* $\cos x$ *or* $\cos^2 x$. Change entirely to cosines and divide out.

E.g., $\dfrac{\sin^4 x}{\cos x} = \dfrac{(1-\cos^2 x)^2}{\cos x} = \sec x - 2\cos x + \cos^3 x.$

This resulting form can be integrated by previous methods.

(VI) *Products* (fractional in form) *in which the sum of the exponents is negative and even.* Transform to powers of $\tan x$ and $\sec x$. Then split off $\sec^2 x\, dx$.

E.g., the product $\sin^8 x \cos^{-10} x$ can be written:

$$\frac{\sin^8 x}{\cos^{10} x} = \left(\frac{\sin x}{\cos x}\right)^8 \left(\frac{1}{\cos x}\right)^2 = \tan^8 x \sec^2 x.$$

Similarly, $\dfrac{1}{\sin x \cos^3 x} = \dfrac{1}{(\tan x)\cos^4 x} = \dfrac{\sec^4 x}{\tan x}.$

Remark. When a combination of $\tan x$ and $\sec x$ is changed into sines and cosines preparatory to using a reduction formula, it is often seen to fall under one of cases (I), (IV), (V) above, and to be most readily integrable without tables.

EXERCISES

Tables may be used whenever helpful.

1. Integrate the following quantities:

(*a*) $\sin^3 \theta \cos^2 \theta \, d\theta$, (*b*) $\cos^5 \theta \, d\theta$, (*c*) $\dfrac{\cos^3 2\theta}{\sqrt{\sin^5 2\theta}} d\theta$,

(*d*) $\tan x \sec^3 x \, dx$, (*e*) $\dfrac{\tan^3 x \, dx}{\sec x}$, (*f*) $\tan^5 3x \, dx$,

(*g*) $\sqrt{\tan t} \sec^4 t \, dt$, (*h*) $\sec^6 t \, dt$, (*i*) $\dfrac{\sec^4 7t \, dt}{\tan^3 7t}$,

(*j*) $\sin^2 \phi \cos^2 \phi \, d\phi$, (*k*) $\cos^4 \phi \, d\phi$, (*l*) $\sin^2 6\phi \, d\phi$,

(*m*) $\dfrac{\sin^4 x}{\cos^2 x} dx$, (*n*) $\dfrac{\cos^2 x}{\sin x} dx$, (*o*) $\dfrac{\sin^2 8x}{\cos 8x} dx$,

(*p*) $\dfrac{dy}{\sin^5 y \cos^3 y}$, (*q*) $\dfrac{dy}{\sin^2 y \cos^4 y}$, (*r*) $\dfrac{dy}{\sin 3y \cos 3y}$.

2. Simplify the following forms and integrate:

(*a*) $\sec^3 x \sin x \, dx$, (*b*) $\operatorname{ctn}^4 \theta \sec^2 \theta \, d\theta$, (*c*) $\dfrac{\sec^3 y}{\tan^4 y} dy$,

(*d*) $\tan^3 x \cos x \, dx$, (*e*) $\dfrac{dt}{\sec t \tan^2 t}$, (*f*) $\dfrac{dt}{\sec^2 t \tan t}$.

3. Find the following integrals:

(*a*) $\int \cos^6 \theta \, d\theta$, (*b*) $\int \sec^5 \phi \, d\phi$, (*c*) $\int \operatorname{ctn}^6 t \, dt$,

(*d*) $\int \dfrac{\sin^2 x}{\cos^3 x} dx$, (*e*) $\int \dfrac{\sec^5 y}{\tan^3 y} dy$, (*f*) $\int \dfrac{\cos^4 x}{\sin^3 x} dx$.

4. Integrate each of the following in two different ways:

(*a*) $\sec^6 \theta \tan^5 \theta \, d\theta$, (*b*) $\sin^3 t \cos^5 t \, dt$, (*c*) $\dfrac{\sin^3 2t}{\cos^5 2t} dt$.

5. Transform and integrate:

(*a*) $\sqrt{1 - \cos 4\theta} \, d\theta$, (*b*) $\sqrt{1 + \cos \theta} \, d\theta$, (*c*) $\sqrt{1 + \sin 6\theta} \, d\theta$.

6. In Ex. 3(*f*) two different procedures gave the results

$$(A)\quad \frac{\cos^3 x - \frac{3}{2}\cos x}{\sin^2 x} - \frac{3}{2}\log(\csc x - \operatorname{ctn} x),$$

$$(B)\quad -\frac{\cos x}{2\sin^2 x} - \cos x - \frac{3}{2}\log\tan\frac{x}{2}.$$

Test the consistency of these two integrals.

7. Like Ex. 6 for the following results in Ex. 4(*a*):

$$(A)\ \frac{\sec^{10}\theta}{10} - \frac{\sec^8\theta}{4} + \frac{\sec^6\theta}{6}, \qquad (B)\ \frac{\tan^6\theta}{6} + \frac{\tan^8\theta}{4} + \frac{\tan^{10}\theta}{10}.$$

8. Find the following integrals by suitable substitutions:

$$(a)\ \int\frac{\cos\theta\,d\theta}{(2+\sin\theta)(1-\sin\theta)^2}, \qquad (b)\ \int\frac{\sec^2\theta\,d\theta}{(1+\tan\theta)(3+\sec^2\theta)}.$$

9. Expand, simplify, and integrate:

(*a*) $\sin(A+\theta)\cos\theta\,d\theta$, (*b*) $\dfrac{\sin(\theta+20°)}{\cos\theta}\,d\theta$,

(*c*) $\dfrac{\sin\theta\,d\theta}{\cos^2(\theta-60°)}$. [Let $\theta = 60° + \phi$.]

10. In studying the energy transmitted through a sphere by periodic oscillations it was necessary to find the integral $\int \pi k \sin^3\theta\,d\theta$. Evaluate this.

11. In deriving a formula giving the position of a comet in its orbit, it is necessary to find $\int r^2\,d\theta$ where $r = a\sec^2\left(\frac{\theta}{2}\right)$. Perform the integration.

12. Find the length of one arch of the cycloid $x = a(\phi - \sin\phi)$, $y = a(1-\cos\phi)$. [Cf. (65), § 81.]

13. For one quadrant of the hypocycloid $x = a\cos^3\phi$, $y = a\sin^3\phi$, find

(*a*) The area A; (*b*) $\bar{x}$ for A; (*c*) I_y for A.

[Express y, x, dx, etc., in terms of ϕ and $d\phi$.]

14. Find the integral $\int\dfrac{dx}{\sin x\cos x}$: (*a*) by tables; (*b*) by VI, § 102; (*c*) by a double-angle formula.

§ 103. Other Rational Forms. Besides powers we have to integrate other expressions which involve trigonometric functions of an angle x in rational form, but do not involve

x in any other way. *Every such expression can be changed into a rational algebraic form.*

For if we put

$$\tan\frac{x}{2}=v, \tag{28}$$

i.e., $x=2\tan^{-1}v$, then

$$dx=\frac{2\,dv}{1+v^2}. \tag{29}$$

Also, as Fig. 59 shows:

$$\sin\frac{x}{2}=\frac{v}{\sqrt{1+v^2}},\quad \cos\frac{x}{2}=\frac{1}{\sqrt{1+v^2}}.$$

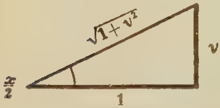


FIG. 59.

Hence by the double-angle formulas, p. 490,

$$\sin x=\frac{2\,v}{1+v^2},\quad \cos x=\frac{1-v^2}{1+v^2}. \tag{30}$$

Then $\tan x=\sin x/\cos x=2\,v/(1-v^2)$. Also we can find values of the reciprocal functions, — all rational.

When formulas (29), (30) are at hand, we can quickly make the change from trigonometric to algebraic form. Then we can integrate by using partial fractions, if we can factor the denominator. When (29), (30) are not at hand, we work them out for ourselves as above. Other substitutions are sometimes simpler than (28), but (28) *always* works.

If different angles are present, they should all be reduced to some one angle before making the substitution. If all the angles are kx instead of x, the process above applies to the various functions of kx. But dx in (29) is slightly changed.

Ex. I. $F=\int\frac{\sin x\,dx}{2\sin x-\cos x+2}.$

Using (28)–(30) we obtain:

$$F=\int\frac{\frac{2\,v}{1+v^2}\cdot\frac{2\,dv}{1+v^2}}{\frac{4\,v}{1+v^2}-\frac{1-v^2}{1+v^2}+2}=\int\frac{4\,v\,dv}{(1+v^2)(3\,v^2+4\,v+1)}.$$

This is rational, and easily reduces to partial fractions:

$$F=\int\left[\frac{1}{v+1}-\frac{9}{5(3\,v+1)}-\frac{2\,v-4}{5(1+v^2)}\right]dv.$$

After integrating this we would replace each v by $\tan\frac{x}{2}$.

EXERCISES

1. Integrate the following quantities:

(a) $\dfrac{\cos x\,dx}{3\sin x+4\cos x}$, (b) $\dfrac{\sin 3\,x\,dx}{\sin 3\,x+\cos 3\,x}$, (c) $\dfrac{dx}{\sin x\,(5+4\cos x)}$,

(d) $\dfrac{\sin x\,dx}{12\sin x+5\cos x}$, (e) $\dfrac{\cos x\,dx}{1+\sin x-\cos x}$, (f) $\dfrac{dx}{\sin x\,(1-\sin x)}$,

(g) $\dfrac{\sin 2\,\theta\,d\theta}{(1+\sin\theta)^2}$, (h) $\dfrac{\sin t\,dt}{1+\cos 2\,t}$, (i) $\dfrac{dy}{1+\cos 4\,y-\sin 4\,y}$.

2. (a)–(d). Derive formulas (58), (59), p. 496, and (70), (71), p. 497, respectively.

3. Derive the integration formula for $\sec x\,dx$:

(a) By using the half-angle substitution (28);

(b) By using a different substitution after writing

$$\sec x\,dx=\frac{\cos x\,dx}{\cos^2 x}.$$

4. Find the integral $\displaystyle\int\frac{dx}{4+\cos^2 x}$ in two ways:

(a) By dividing numerator and denominator by $\cos^2 x$ and expressing the denominator in terms of $\tan x$;

(b) By substituting for the number 4 the expression $4\sin^2 x+4\cos^2 x$ and then using the tables.

5. To what sort of fractional integrand would the half-angle substitution (28) lead in Ex. 4?

6. (a), (b). Like Ex. 4 (a), (b) for the integral

$$\int\frac{dx}{25-16\sin^2 x}.$$

(c) Also integrate by using the half-angle substitution (28).

§ 104. Several Angles. The product of the sine or cosine of one variable angle by the sine or cosine of another is integrated by first changing the product to a *sum*. This

is done by one of the following formulas (obtainable from *Intro.*, § 285, or by expanding the right members):

$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)], \tag{31}$$

$$\cos A \cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)], \tag{32}$$

$$\sin A \cos B = \tfrac{1}{2}[\sin(A-B) + \sin(A+B)]. \tag{33}$$

Integrals of products found in this way are given in the Tables, p. 497. But it is well to know formulas (31)–(33) as they help also with more complicated forms.

Ex. I. $F = \int \cos 2x \cos 5x \sin 6x \, dx.$

First using (32) for the two cosines, we get

$$F = \tfrac{1}{2}\int [\cos(-3x) + \cos 7x] \sin 6x \, dx.$$

But $\cos(-3x) = \cos 3x$. The products $\sin 6x \cos 3x$ and $\sin 6x \cos 7x$ can now be integrated by Tables. Or, applying (33) to each:

$$F = \tfrac{1}{4}\int [\sin 3x + \sin 9x + \sin(-x) + \sin 13x] \, dx.$$

Also, $\sin(-x) = -\sin x$. Each term is now easily integrated.

§ 105. Non-Integrable Forms. Many trigonometric forms are not integrable in the elementary sense: although integrals exist they are not expressible in terms of the elementary functions.

Some apparently simple integrands of this sort, which arise in practical problems, are:

$$\sin(x^2)dx, \qquad \frac{\sin x}{x}dx, \qquad \sqrt{\sin x}\,dx.$$

Later we shall see (§§ 176–180) how to approximate the integrals of such forms to any desired degree of accuracy.

As a matter of fact, that is all we do when we find such an integral as

$$\int \frac{dx}{x} = \log x,$$

and then look up the logarithm in a table. But we profit, of course, by the known properties of logarithms and by the existence of tables.

EXERCISES

1. Finish Ex. I, § 104. Also perform the integration from the start by first combining $\sin 6x$ and $\cos 2x$.

2. Integrate the following quantities, using tables if helpful:

(*a*) $\sin x \sin 3x \cos 9x\, dx$, (*b*) $(1+\sin x)\cos 4x \cos 6x\, dx$,

(*c*) $\cos 2x \cos 4x \sin^2 5x\, dx$, (*d*) $\cos x \cos 7x \cos \frac{x}{2} \cos \frac{3x}{2}\, dx$,

(*e*) $\sin 4x\,(1+\cos 2x)^2\, dx$, (*f*) $\sin 2x \sin 4x \sin 7x \sin 9x\, dx$.

3. Derive integration formulas (78)–(80), p. 497.

4. If $y = A + B\cos x + C\cos 2x + D\cos 3x$, find

$$\int_0^\pi y\cos 5x\, dx, \qquad \int_0^\pi y\cos 2x\, dx, \qquad \int_0^\pi y\, dx.$$

5. If $y = A\sin x + B\sin 2x + C\sin 3x$, find

$$\int_0^\pi y\sin 7x\, dx, \qquad \int_0^\pi y\sin 2x\, dx.$$

6. Find the integrals

(*a*) $\displaystyle\int \frac{\sin x\, dx}{(1+\cos x)(1+\cos^2 x)}$, (*b*) $\displaystyle\int \frac{\sin 8x\, dx}{1+\sin 4x}$, (*c*) $\displaystyle\int \frac{(3+\cos 2x)dx}{4\sin x + 3\cos x}$.

§ 106. Substitution in Algebraic Radicals. Any quadratic radical can be rationalized by a trigonometric substitution. The three basic types are as follows.

Radical	*Substitution*
$\sqrt{a^2-x^2}$	$x = a\sin\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$

If necessary to return to the variable x after integrating, this can be done by drawing a triangle, as in Fig. 60 below.

Any quadratic radical can be reduced to one of these three types, possibly times a constant. *E.g.*, $\sqrt{3-2x^2}$ could be written $\sqrt{2}\sqrt{\frac{3}{2}-x^2}$; and we could put $x = \sqrt{\frac{3}{2}}\sin\theta$. Again, $\sqrt{x^2-6x+5}$ could be written $\sqrt{(x-3)^2-4}$, or $\sqrt{y^2-4}$ if we let $x-3=y$. Then we could put $y = 2\sec\theta$.

When using tables, these trigonometric substitutions are needed only in somewhat rare and complicated cases. But

when working without tables, they provide the simplest means of handling certain cases, — notably those in which algebraic substitutions lead to expressions of the type

$$\frac{dt}{(t^2+a^2)^n}, \tag{34}$$

for the integration of which we have thus far relied upon reduction formulas.

Ex. I. Find without tables: $F=\int\sqrt{4-x^2}\,dx.$

Putting $x=2\sin\theta$, the radical becomes $\sqrt{4(1-\sin^2\theta)}=2\cos\theta$. Also $dx=2\cos\theta\,d\theta$. Then

$$F=\int 4\cos^2\theta\,d\theta=2\int(1+\cos 2\,\theta)d\theta=2\,\theta+\sin 2\,\theta+C. \tag{35}$$

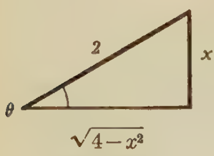


FIG. 60.

Or, reducing $\sin 2\,\theta$: $F=2\,\theta+2\sin\theta\cos\theta+C$.

Since $\sin\theta=x/2$, then by Fig. 60, $\cos\theta=\sqrt{4-x^2}/2$.

$$\therefore\quad F=2\sin^{-1}\left(\frac{x}{2}\right)+\frac{x}{2}\sqrt{4-x^2}+C. \tag{36}$$

Remarks. (I) In finding a definite integral we would change limits instead of going back to x.

(II) Possible *algebraic* substitutions in the case above would be:

(1) $\sqrt{4-x^2}=(2+x)t$, as in § 99;

or, after factoring x out,

(2) $\sqrt{4\,x^{-2}-1}=t$, as in § 97.

Either of these would produce troublesome fractions of the type (34) above. And while these could now be integrated by letting $t=a\tan\theta$, the labor would be much greater than that above.

§ **107. Trigonometric Summary.** The procedure for integrating trigonometric forms with or without tables may now be summarized. Do not memorize the various methods as rules, or use the tabulation mechanically for reference,

but rather *see* from numerical illustrations why each is suitable.

TWO OR MORE ANGLES

Each a multiple of x

In general: Reduce to functions of the single angle x.
Special products: Change products into sums. (§ 104.)

ANY RATIONAL FORM

Involving a single angle x

In general: Change to rational algebraic, by $\tan \frac{x}{2} = v$.
Powers, etc.: See quicker methods below.

POWERS AND THEIR PRODUCTS AND QUOTIENTS

All varieties can be integrated without tables by the methods (A)–(F) below or the modifications printed under each in small type. In several cases these are quicker than reduction formulas even when tables are available. (Cf. *Note* below.)

In the following list, sin x may be interchanged with cos x, also tan x and sec x with ctn x and csc x.

Unless otherwise stated, m and n may be $+$, $-$, or zero; and either even or odd. But $2\,m$ is even and $2\,m+1$ odd.

(A) $\sin^{2m+1} x \cos^n x\, dx$: Split off $-\sin x\, dx$.

If m is $-$, first multiply and divide by $\sin x$. Or, better, if n is odd and $+$, split off $\cos x\, dx$; or, if n is odd and $-$, change to Case (E). Cf. (C) below.

(B) $\sin^{2m} x \cos^{2n} x\, dx [m+n>0]$: Use the double-angle formulas.

When n is $-$, change entirely to cosines and divide out.

(C) $\sin^{2m} x \cos^{2n} x\, dx [m+n<0]$: Change to Case (E).

Dividing $\sin^{2m} x$ by $\cos^{2m} x$, the form becomes $\tan^{2m} x \sec^{-2(m+n)} x\, dx$.

(*D*) $\tan^{2m+1} x \sec^n x\, dx$: Split off $\tan x \sec x\, dx$.

When m is $-$, first multiply and divide by $\tan x$. Or, better, if n is even and $+$, use Case (*E*); or, if n is even and $-$, change to sines and cosines.

(*E*) $\tan^m x \sec^{2n} x\, dx\, [n>0]$: Split off $\sec^2 x\, dx$.

When $n=0$ and m is even, first change to secants (or cosecants when m is $-$); but if m is odd, use Case (*D*). When n is $-$, use Case (*D*) if m is odd and $+$, otherwise change to sines and cosines.

(*F*) $\tan^{2m} x \sec^{2n+1} x\, dx$: Change to sines and cosines.

Note. These methods are usually quicker than reduction formulas in Case (*C*); also (*A*) and (*D*) if m is $+$; (*E*) if n is $+$, (*F*) if n is $-$, and (*B*) if m and n are small.

EXERCISES

1. Integrate by using a trigonometric substitution:

(*a*) $\dfrac{x^3\, dx}{\sqrt{1-x^2}}$, (*b*) $\dfrac{dx}{x^3\sqrt{x^2-4}}$, (*c*) $\dfrac{x^2\, dx}{\sqrt{9-x^2}}$,

(*d*) $\dfrac{x^3\, dx}{\sqrt{16+x^2}}$, (*e*) $\dfrac{dx}{(a^2-x^2)^{\frac{3}{2}}}$, (*f*) $\dfrac{dx}{(x^2-a^2)^{\frac{3}{2}}}$,

(*g*) $\dfrac{x^3\, dx}{(x^2-1)^{\frac{5}{2}}}$, (*h*) $\dfrac{x^3\, dx}{(4+x^2)^{\frac{3}{2}}}$, (*i*) $\dfrac{dx}{(x^2+4)^2}$.

2. (*a*)–(*i*). Check each result in Ex. 1 (*a*)–(*i*), respectively, either by using the tables or by some other method.

3. Calculate in two different ways:

$$\int_0^2 \frac{dx}{x+\sqrt{4-x^2}}.$$

4. Derive formula (76), p. 497, in the special case

$$\int \sqrt{25-9\sec^2\theta}\, d\theta.$$

[Hint: Let the radical equal u; and later put $u=4\cos t$.]

5. Verify formula (77), p. 497, by differentiating the right-hand member.

6. Integrate by inspection or after slight transformations:

(a) $\dfrac{\sec^2\theta\,d\theta}{\sqrt{1+\tan\theta}}$, (b) $\dfrac{\cos\theta\,d\theta}{10-\cos^2\theta}$, (c) $\dfrac{\sec\theta\tan\theta\,d\theta}{\sqrt{\tan^2\theta+5}}$.

7. Find the integrals:

(a) $\displaystyle\int\frac{\sin^3 x\,dx}{\cos^6 x}$, (b) $\displaystyle\int\frac{\sin 3\,x\,dx}{2+\cos 3\,x}$, (c) $\displaystyle\int\frac{\sin 2\,x\,dx}{1-\cos 2\,x}$,

(d) $\displaystyle\int\sec^8 2\,\theta\,d\theta$, (e) $\displaystyle\int\cos^5 t\,dt$, (f) $\displaystyle\int\sin^2 4\,x(1+\cos x)dx$.

PART IV. INTEGRATION BY PARTS

§ **108. The Idea.** The formula for the differential of a product gives rise to a very powerful method of integration, by which we can handle many transcendental functions, products, etc.; and can derive the various reduction formulas.

Since
$$d(uv)=udv+vdu,$$
$$\therefore\ uv=\int udv+\int vdu.$$

Often one of these integrals is much simpler than the other and can be used to find the other:

$$\int udv=uv-\int vdu. \tag{37}$$

Regard any troublesome integrand as udv. That is, split it into two factors, calling one u and the other dv, the differential of some function. By integrating dv and differentiating u separately, we get v and du. If, now, vdu is much easier to integrate than the original udv, we can use (37) advantageously.

This process is called "Integration by Parts."

Ex. I. $F=\int x^4 \log x\, dx.$

Try $u=\log x,\qquad dv=x^4\, dx.$

Then $du=\dfrac{dx}{x},\qquad v=\frac{1}{5}x^5.$

$$\therefore\ F=uv-\int v du=\tfrac{1}{5}x^5\log x-\tfrac{1}{5}\int x^4\, dx$$
$$=\tfrac{1}{5}x^5[\log x-\tfrac{1}{5}]+C.$$

Remark. We can usually see in advance whether a proposed choice of u and dv is promising, by thinking a moment as to what sort of expressions would result from differentiating the contemplated u and integrating the dv. Anyhow, a poor choice will soon become apparent, and we can try another.

§ 109. **Integrals from Equations.** It is often necessary to integrate by parts two or more times in succession. In fact, the process may never bring us to a known integral; but in some cases we arrive at an equation in which the given unknown integral F reappears, and from which we can find F by transposing and solving algebraically.

Ex. I. $F=\int e^{2x}\sin 3x\, dx.$

Taking $u=e^{2x}$ and $dv=\sin 3x\, dx$, we get $du=2e^{2x}\, dx$ and $v=-\frac{1}{3}\cos 3x$. Hence

$$F=-\tfrac{1}{3}e^{2x}\cos 3x+\tfrac{2}{3}\int e^{2x}\cos 3x\, dx. \tag{38}$$

Again taking $u=e^{2x}$, with $dv=\cos 3x\, dx$, we get

$$F=-\tfrac{1}{3}e^{2x}\cos 3x+\tfrac{2}{3}[\tfrac{1}{3}e^{2x}\sin 3x-\tfrac{2}{3}\int e^{2x}\sin 3x\, dx]. \tag{39}$$

Now this last integral is F itself. Hence

$$F=-\tfrac{1}{3}e^{2x}\cos 3x+\tfrac{2}{9}e^{2x}\sin 3x-\tfrac{4}{9}F. \tag{40}$$

Transposing $\frac{4}{9}F$ gives $\frac{13}{9}F$. Dividing through by $\frac{13}{9}$ and factoring:

$$F=\frac{e^{2x}}{13}[2\sin 3x-3\cos 3x]. \tag{41}$$

EXERCISES

1. Find without tables the following integrals:

(*a*) $\int x^3 \log x\, dx$, (*b*) $\int x \sin 2x\, dx$, (*c*) $\int x^2 \cos 3x\, dx$,

(*d*) $\int \sin^{-1}\left(\frac{4}{x}\right) dx$, (*e*) $\int x\, e^{5x}\, dx$, (*f*) $\int e^{3x} \cos 4x\, dx$.

2. Similarly derive the integration formulas on p. 498 for

(*a*) $\sin^{-1} x\, dx$, (*b*) $\tan^{-1} x\, dx$, (*c*) $x^n a^{mx}\, dx$,
(*d*) $\operatorname{vers}^{-1} x\, dx$, (*e*) $x^n \log x\, dx$, (*f*) $e^{kx} \sin nx\, dx$.

3. Find the following integrals, using tables wherever helpful:

(*a*) $\int x \sin^{-1} x\, dx$, (*b*) $\int \frac{\tan^{-1} x}{x^2}\, dx$,

(*c*) $\int \log x \cdot \cos^{-1} x\, dx$, (*d*) $\int \sqrt{1+x^2} \log (x+\sqrt{1+x^2})\, dx$,

(*e*) $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$, (*f*) $\int \tan^{-1}\sqrt{\frac{x}{a}}\, dx$.

[Suggestion: In (*f*), let $x = at^2$. Also show that (*e*) is reducible to (*f*).]

4. An integral needed in actuarial science is: $\int_0^1 t(1.035)^{-t}\, dt$. Evaluate by § 108. Check by tables.

5. Find $\int \frac{dt}{(1+t^2)^2}$ by writing the numerator as a difference, $(1+t^2)\, dt - t^2\, dt$, and integrating by parts the second integral so obtained.

§ 110. Derivation of Reduction Formulas: Trigonometric Forms. Most of our reduction formulas, pages 496–97, are obtained by integrating by parts and solving the resulting equation algebraically. Some typical illustrations follow.

Ex. I. $F = \int \sec^n x\, dx$, $n \neq 1$.

Split off $\sec^2 x\, dx$ as dv, taking the rest as u: $u = \sec^{n-2} x$.
Then $du = (n-2) \sec^{n-2} x \tan x\, dx$, $v = \tan x$.

$$\therefore \qquad F = uv - (n-2)\int \sec^{n-2} x \tan^2 x\, dx. \tag{42}$$

Replacing $\tan^2 x$ by $\sec^2 x - 1$, the latter integral splits into two, one of which is F itself.

$$F = uv - (n-2)[F - \int \sec^{n-2} x \, dx]. \tag{43}$$

Solving for F, and inserting the values of u and v:

$$F = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx. \tag{44}$$

This formula will reduce the given exponent by 2 if $n \neq 1$. And, if $n = 1$, it is not needed.

Ex. II. $F = \int \sin^m x \cos^n x \, dx$: to reduce n.

Take $\sin^m x \cos x \, dx$ as dv, and $\cos^{n-1} x$ as u. Then

$$du = -(n-1) \cos^{n-2} x \sin x \, dx, \quad v = \frac{\sin^{m+1} x}{m+1}.$$

$$\therefore \qquad F = uv + \frac{n-1}{m+1} \int \sin^{m+2} x \cos^{n-2} x \, dx. \tag{45}$$

Now split off $\sin^2 x$ $(= 1 - \cos^2 x)$, and we get

$$F = uv + \frac{n-1}{m+1} \left[\int \sin^m x \cos^{n-2} x \, dx \quad - F \right]. \tag{46}$$

Collecting and solving for F gives finally:

$$F = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx. \tag{47}$$

This fails if $m + n = 0$. But then $n = -m$, and the original $\sin^m x \cos^n x \, dx$ is simply $\tan^m x \, dx$, which can be reduced otherwise.

To raise n, when negative, solve (47) for the final integral in terms of F; and replace n by $n+2$ throughout. An algebraic case of similar character is worked out below.

§ 111. Algebraic Forms. The basic algebraic reduction formulas (27) – (30), p. 494, are derived by methods like those above.

Ex. I. $F=\int x^m(ax^n+b)^p\,dx$: to reduce m.

Take $(ax^n+b)^p\,x^{n-1}\,dx$ as dv, and x^{m-n+1} as u. Then

$$du=(m-n+1)\,x^{m-n}\,dx, \quad v=\frac{1}{na(p+1)}\,(ax^n+b)^{p+1}.$$

$$\therefore \qquad F=uv-\frac{m-n+1}{na(p+1)}\int x^{m-n}\,(ax^n+b)^{p+1}\,dx. \tag{48}$$

Split off the first power of (ax^n+b) and combine with x^{m-n}:

$$\begin{aligned} x^{m-n}\,(ax^n+b)^{p+1} &= (ax^m+bx^{m-n})(ax^n+b)^p \\ &= ax^m(..)^p+bx^{m-n}(..)^p. \end{aligned} \tag{49}$$

$$\therefore F=uv-\frac{m-n+1}{na(p+1)}\left[aF+b\int x^{m-n}\,(ax^n+b)^p\,dx\right]. \tag{50}$$

Multiplying through by $na\,(p+1)$, transposing, and collecting:

$$\begin{aligned} a(m+np+1)F &= x^{m-n+1}\,(ax^n+b)^{p+1} \\ &\quad -(m-n+1)b\int x^{m-n}(ax^n+b)^p\,dx. \end{aligned} \tag{51}$$

This at once gives formula (29), p. 494, for reducing the outside exponent by n.

The formula fails if $m+np+1=0$; but in that case it is not needed. For factoring out $(x^n)^p$ from the original parenthesis, as in § 97, will make the outside exponent $m+np$, or -1. And -1 differs from the exponent needed in the differential of the new inside power by a multiple of $-n$.

Ex. II. Derive formula (30), p 494, for raising the outside exponent.

Solving (51) for the last integral in terms of F, and denoting the quantity ax^n+b by Q, we get

$$\int x^{m-n}\,Q^p\,dx=\frac{x^{m-n+1}\,Q^{p+1}}{b(m-n+1)}-\frac{a(m+np+1)}{b(m-n+1)}\int x^m\,Q^p\,dx. \tag{52}$$

Now let $m-n=k$, or $m=k+n$; and we have

$$\int x^k Q^p\,dx=\frac{x^{k+1}\,Q^{p+1}}{b(k+1)}-\frac{a(k+n+np+1)}{b(k+1)}\int x^{k+n}\,Q^p\,dx. \quad (53)$$

This agrees with (30), p. 494, with k here in place of m there. It is immaterial what letter we use in such a formula.

EXERCISES

1. (*a*) Carry out in detail the steps in Ex. I, § 110, in the special case where $n=7$. Likewise for the following cases:
(*b*) Ex. II, § 110, using $m = 4$, $n = 8$;
(*c*) Ex. I, § 111, using $m = 8$, $n = 2$, $p = 5$;
(*d*) Using the result of Ex. II, § 110, for the case $m=4$, $n=-6$, solve for the final integral in terms of the original, thus expressing

$$\int \sin^4 x \cos^{-8} x\,dx \text{ in terms of } \int \sin^4 x \cos^{-6} x\,dx.$$

2. In a similar manner derive formulas expressing

(*a*) $\int \sin^6 \theta\,d\theta$ in terms of $\int \sin^4 \theta\,d\theta$;

(*b*) $\int x^7(ax^2+b)^{\frac{3}{2}}\,dx$ in terms of $\int x^7(ax^2+b)^{\frac{1}{2}}\,dx$;

(*c*) $\int x^7(ax^2+b)^{\frac{1}{2}}\,dx$ in terms of $\int x^5(ax^2+b)^{\frac{1}{2}}\,dx$;

(*d*) $\int x^7(ax^2+b)^{-\frac{3}{2}}\,dx$ in terms of $\int x^7(ax^2+b)^{-\frac{5}{2}}\,dx$, and reverse.

3. Integrate the following quantities, using tables if helpful.

(*a*) $(1+x^2)\log(x+\sqrt{1+a^2+x^2})dx$, (*b*) $\sin^{-1}\dfrac{4}{\sqrt{25-x^2}}\,dx$,

(*c*) $(25-y^2)\sin^{-1}\left(\dfrac{3}{\sqrt{25-y^2}}\right)dy$, (*d*) $x\sin^{-1}\sqrt{\dfrac{x}{a+x}}\;dx$.

[Hint: In (*a*) – (*c*) after integrating once by parts, use a trigonometric substitution and divide out.]

4. The intensity of illumination at the edge of a certain area is

$$H=\int_a^b 2\,J\rho \arccos\left(\frac{a}{\rho}\right)d\rho.$$

Carry out the integration for a special case in which J is constant.

§ 112. Integration Procedure Summarized. As a first step in integrating we should always examine the integrand carefully and ask ourselves several questions: Would it be a type form if we called some quantity u? If so, exactly what type? Could it be found in the tables? Would a reduction formula help?

When necessary or advantageous to transform, mentally classify the given form and proceed accordingly.

(*I*) *Rational Algebraic Forms.* Separate into partial fractions. The only non-type fractions obtained are those with trinomial denominators (which we can change into binomials by completing the square), and

$$\frac{dx}{(x^2+a^2)^n} \qquad n>1.$$

This last can be integrated by letting $x = a \tan \theta$, and then using double-angle formulas; or by a reduction formula.

(*II*) *Irrational Algebraic Forms.* Let any linear radical equal t; also any higher radical if there is a suitable power of x outside, — either as given or after factoring out a power. When a quadratic radical appears with a polynomial outside in the denominator, let the radical equal $(t - x)$ or one factor times t, as seems best.

(*III*) *Trigonometric Forms.* Proceed as outlined in § 107.

(*IV*) *Miscellaneous Combinations.* Try integration by parts, and devices like those used in deriving the reduction formulas.

Finally, if necessary, resort to approximation as explained later. (§§ 176–180.)

EXERCISES ON CHAPTER IV

Integrate the following quantities by any method available:

1. $\dfrac{(x+2)dx}{x\sqrt{x^2-5x+4}}$. **2.** $\dfrac{x^4\,dx}{(x-1)^3}$. **3.** $e^{4x} \cos x\,dx$.

4. $\dfrac{(\cos x+3)dx}{2-\sin x}.$ **5.** $\dfrac{dx}{x\sqrt{x^6-1}}.$ **6.** $\sin^8\theta\cos\theta\,d\theta.$

7. $\sin^2 x\cos^5 x\,dx.$ **8.** $e^{2x}x^4\,dx.$ **9.** $\cos^{-1}x\,dx.$

10. $e^{4x}\cos^2 x\,dx.$ **11.** $x^3\sin x\,dx.$ **12.** $\dfrac{\sin^8\theta}{\cos^2\theta}d\theta.$

13. $\dfrac{\sqrt{x^2-6x+2}}{x+5}dx.$ **14.** $\dfrac{(2x+5)dx}{\sqrt{x^2-5x+4}}.$ **15.** $\dfrac{3x^5\,dx}{(x^6-4x^3)^{\frac{1}{2}}}.$

16. $\dfrac{dx}{\sin x+2\cos x}.$ **17.** $\sec^4 x\,dx.$ **18.** $\dfrac{\sin^4\theta\,d\theta}{\cos\theta}.$

19. $\dfrac{(2x+5)dx}{x^4-2x^2-9x+10}.$ **20.** $\dfrac{x^2\,dx}{x^6-64}.$ **21.** $\dfrac{\sqrt{x^2-6x+10}}{2x+5}dx.$

22. $\dfrac{dx}{\sin x+2\cos x+3}.$ **23.** $\dfrac{\sqrt{x^4+1}}{x}dx.$ **24.** $\dfrac{\cos^2\theta}{\sin^8\theta}d\theta.$

25. $\dfrac{dx}{x\sqrt{x^2-x+1}}.$ **26.** $\dfrac{\sqrt{x+4}}{x+1}dx.$ **27.** $\dfrac{\sin^3\theta}{\cos^8\theta}d\theta.$

28. $\log x\arcsin x\,dx.$ **29.** $\sec^3 x\,dx.$ **30.** $\cos 2x\cos 5x\,dx.$

31. $\dfrac{x\,\mathrm{vers}^{-1}x\,dx}{\sqrt{2x-x^2}}.$ **32.** $\dfrac{dx}{x^3-5}.$ **33.** $\dfrac{dx}{(x^2+3x+4)^3}.$

34. $\dfrac{\cos\theta\,d\theta}{\sin 2\theta}.$ **35.** $\dfrac{\cos 2\theta\,d\theta}{\sin\theta}.$ **36.** $\dfrac{\sin 3t\,dt}{\cos t}.$

CHAPTER V

DOUBLE AND TRIPLE INTEGRATION

PART I. DOUBLE INTEGRATION IN A PLANE

§ 113. **Successive Partial Integration.** In many scientific problems it is necessary to integrate twice in succession. Sometimes the integrations are both performed with respect to the same variable. (*E.g.*, in finding the distance traveled by an object moving with a known acceleration, we integrate twice with respect to t.) In some other problems, to be considered now, there are two independent variables x and y; and we have to integrate once with respect to each.

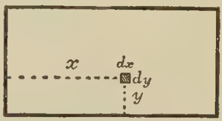


Fig. 61 *a*.

Ex. I. The load per square foot on a rectangular floor 20 ft. by 10 ft. varies thus with the distances x ft. and y ft. from one end and side:

$$w = 60 - .06\,x^2 - .12\,y^2.$$

Find the total load on the floor.

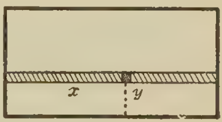


Fig. 61 *b*.

We cannot use a strip across the floor in either direction, for the load per sq. ft. varies much along any such strip.

Consider a tiny rectangular piece of floor, practically a point (x, y), of dimensions dx and dy. (Fig. 61 *a*.) The load on this tiny area $dx\,dy$ is

$$(60 - .06\,x^2 - .12\,y^2)\,dx\,dy. \qquad (1)$$

To find the load on a lengthwise strip, like that in Fig. 61 *b*, we sum up (integrate) these tiny weights, for all values

of x from 0 to 20. Along such a strip y is constant; and it must be treated as such in this integration. *E.g.*, integrating $.12\,y^2$ with respect to x will give $.12\,y^2x$.

$$\therefore \text{Load on strip} = \left[\int_0^{20}(60-.06\,x^2-.12\,y^2)dx\right]dy$$
$$= \left[60\,x-.02\,x^3-.12\,y^2x\right]_0^{20}dy$$
$$= (1040-2.4\,y^2)\,dy. \tag{2}$$

To find the total load on the floor, we now sum up (integrate) these strip-loads for all such strips, from $y = 0$ to $y = 10$:

$$L = \int_0^{10}(1040-2.4\,y^2)dy = 1040\,y - .8\,y^3 = 9{,}600.$$

Notation. These steps are conveniently symbolized by writing

$$L = \int_0^{10}\int_0^{20}(60-.06\,x^2-.12\,y^2)dx\,dy. \tag{3}$$

This means that we first integrate $(60-.06\,x^2-.12\,y^2)$ with respect to x from $x=0$ to $x=20$, and then integrate the result with respect to y from $y=0$ to $y=10$. The order of integrations is indicated *from the inside outward*, both as to the integral signs and as to the differentials dx and dy.

In the example above it would be possible to integrate in the reverse order, — first with respect to y to find the load on a *crosswise* strip, and then with respect to x, to sum up for all such strips:

$$L = \int_0^{20}\int_0^{10}(60-.06\,x^2-.12\,y^2)dy\,dx.$$

§ **114. Variable Limits.** The limits for the first of two integrations are not always absolute constants. Thus, for the half-floor shown in Fig. 62, the value to which x would run in the first integration would be different for different strips, depending upon y for each strip. The diagonal has inter-

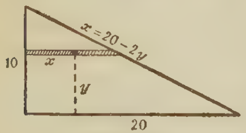


FIG. 62.

cepts 20 and 10 on the X- and Y-axes. Hence by § 3 its equation is

$$\frac{x}{20}+\frac{y}{10}=1, \qquad \text{or} \qquad x=20-2\,y. \tag{4}$$

This latter value is the limit to which x runs for any lengthwise strip.

$$\therefore \qquad L=\int_0^{10}\int_0^{20-2y}(60-.06\,x^2-.12\,y^2)dx\,dy. \tag{5}$$

If we integrated first with respect to y, to get a crosswise strip, y would not run from 0 to 10 but only to a value depending upon x, viz. $y=10-\frac{1}{2}x$. But to get all strips x would then run from 0 to 20:

$$L=\int_0^{20}\int_0^{10-\frac{1}{2}x}(60-.06\,x^2-.12\,y^2)dy\,dx. \tag{6}$$

Study carefully the limits used in (5) and (6); and notice that, although 20 and 10 are the largest values of x and y anywhere in the figure, the first limit of integration in either (5) or (6) is something very different. We must use *not a strip along the boundary* but a typical *general strip running through the area* in question, anywhere inside, as in (5) or (6) above.

§ 115. Critical Note. Accurately speaking, we should say above that the load on any small rectangle $\Delta y\,\Delta x$ is approximately

$$w\,\Delta y\,\Delta x,$$

where the value of w is taken at the middle of the rectangle. If there are m such rectangles in a given crosswise strip, the load on that strip is approximately the limit as $\Delta y \to 0$ of the sum

$$[w_1\,\Delta y+w_2\,\Delta y+\cdots+w_m\,\Delta y]\Delta x.$$

And if there are n such strips, the total load is the limit, as $\Delta x \to 0$, of the sum of n such strip-approximations.

But we can show (Appendix, p. 487) that, as in the case of a single integral, the limit of the sum in question is the definite (double) integral set up above. By regarding the load as a *sum*, when in reality it is the limit of that sum, and then using integration to "sum up," which really

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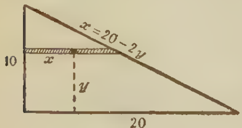

FIG. 62.

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But we can show (Appendix, p. 487) that, as in the case of a single integral, the limit of the sum in question is the definite (double) integral set up above. By regarding the load as a *sum*, when in reality it is the limit of that sum, and then using integration to "sum up," which really

gives the *limit* of the sum, we get a strictly correct value. The same will be true in what follows.

The discussion of this matter in the Appendix can be read most advantageously at the end of the chapter.

§ 116. Ultimate Elements of Area. Any plane area may be regarded as composed of very narrow strips, straight or curved, uniform in width or tapering, so chosen that along each strip one coördinate, x or y, r or θ, may be considered as constant. Each strip in turn is composed of tiny "elements," almost points, of such a shape as to fit together perfectly, end to end or side by side, to form the strip.

(*A*) *Rectangular Coördinates.* The ultimate element is a rectangle $dx\,dy$. A horizontal row of these forms a strip of width dy, along which y is constant. A vertical row forms a strip of width dx, along which x is constant.

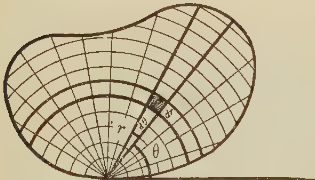


Fig. 63.

(*B*) *Polar Coördinates.* Choose an element of the shape shown in Fig. 63. All elements that have a fixed radius vector r will form a narrow circular ring of width dr, — or a portion of such a ring. All that have a fixed angle θ will form a pointed strip between radii vectores whose included angle is $d\theta$ (radians).

The dimensions of this element are dr and $r\,d\theta$. (Cf. § 81.) And, as adjacent sides meet at right angles, we regard the element as a rectangle, of area

$$r\,d\theta\,dr, \qquad \text{or} \qquad r\,dr\,d\theta. \tag{7}$$

In any problem concerning plane areas or flat plates we choose either element, $dx\,dy$ or $r\,dr\,d\theta$, according as the integration and limits would be simpler in rectangular or in polar coördinates. The latter, being probably less familiar, will be illustrated more fully in what follows.

EXERCISES

1. Evaluate each of these integrals:

(a) $\displaystyle\int_0^{10}\int_0^{\sqrt{100-x^2}} x\,y\,dy\,dx,$ (b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{2\sin\theta} r^4\cos\theta\,dr\,d\theta.$

2. Find the load on a rectangular floor with vertices (0, 0), (40, 0), (40, 20) and (0, 20), if the loading per sq. ft. is $w=20-.1\,x-.2\,y$. Find also the load on the triangular half of the floor which has the vertices (0, 0), (40, 0), (0, 20).

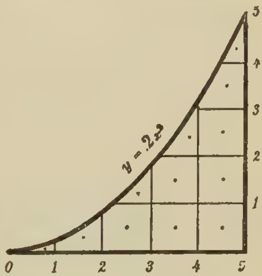
3. If the loading per sq. ft. is $w=40+.1\,x-.2\,y$, find the total load on a triangular floor with each following set of vertices:

(a) (0, 0), (50, 0), (50, 30); (b) (0, 0), (50, 0), (0, 30);
(c) (0, 0), (0, 30), (50, 30); (d) (50, 30), (50, 0), (0, 30).

4. A floor has the shape bounded by $y=x^2$ and $y=25$; and the loading per sq. ft. is $w=10+x+y$. Find the total load.

5. In Ex. 4 find the torque of the load about the X-axis.

6. Plot the area under $y=.2\,x^2$ from $x=0$ to $x=5$. Find the total load on this area if the loading per unit area is $w=x+y$.



7. In Ex. 6 check roughly by estimating the load on each complete square or other portion, using values of w for points somewhat centrally located in the several portions, — as in the adjacent figure.

8. A rectangular plate 10 in. by 4 in. which weighs 3 lb./sq. in. is about to slide on a horizontal table. The coefficient of friction (*i.e.*, ratio of friction to weight) is variable: $f=.1+.002\,x+.005\,y$. Find the total friction F.

9. Find the load on a semi-circular floor of radius 10 ft. if the loading varies thus with the distance r from the center of the circle: $w=50-.3\,r$.

10. Like Ex. 9 for a quarter-circular floor of radius 20 ft., if $w=60-.6\,r$.

§ 117. Area and Mass of a Plate. Consider a thin, flat plate having the shape of the larger segment of a circle of diameter 20 cm., cut off by a line 3 cm. from the center. And suppose that the surface density D at any point, or mass

per sq. cm., varies inversely as the distance r from the missing end of the diameter of symmetry, say $D=k/r$. Let us calculate the total mass of the plate.

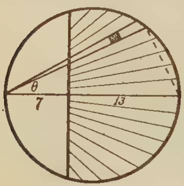


FIG. 64.

$$\text{Mass of element} = D\, r\, dr\, d\theta = k\, dr\, d\theta.$$

Integrating first with respect to r, with θ constant, gives the mass of a tapering strip, starting at the straight line and running to the circle. Then integrating with respect to θ for all such strips gives the entire mass.

At the line, r is the hypotenuse of a right triangle and 7 the side adjacent to θ, whence $r \cos\theta = 7$; *i.e.*,

$$r = 7 \sec\theta. \tag{8}$$

At the circle r is the adjacent side and 20 the hypotenuse, whence

$$r = 20 \cos\theta. \tag{9}$$

These values of r are the limits for the first integration.

To get the final upper limit for θ, equate the values of r for the line and circle, and solve:

$$20 \cos\theta = \frac{7}{\cos\theta}, \qquad \cos\theta = \sqrt{\tfrac{7}{20}},$$

i.e., $\theta = \arccos\sqrt{\frac{7}{20}}$, or $\cos^{-1}\sqrt{\frac{7}{20}}$. Integrating from $\theta = 0$ to this value gives half of the required mass M.

$$\therefore\ M = 2\int_0^{\cos^{-1}\sqrt{\frac{7}{20}}}\int_{7\sec\theta}^{20\cos\theta} k\, dr\, d\theta$$

$$= 2\,k\Big[20\sin\theta - 7\log(\sec\theta + \tan\theta)\Big]_0^{\cos^{-1}\sqrt{\frac{7}{20}}}$$

When $\cos\theta = \sqrt{\frac{7}{20}}$, we find $\sin\theta = \sqrt{\frac{13}{20}}$, $\tan\theta = \sqrt{\frac{13}{7}}$, etc.

$$\therefore\ M = 2\,k\Big[20\sqrt{\tfrac{13}{20}} - 7\log\Big(\sqrt{\tfrac{20}{7}} + \sqrt{\tfrac{13}{7}}\Big)\Big].$$

To find the *area* of the above plate instead of its mass, we could integrate $r\, dr\, d\theta$ instead of $k\, dr\, d\theta$, using the *same limits* as above. This procedure is often useful in finding areas between plane curves; but, in the case above, a rectangular element $dy\, dx$ would be simpler, or a single integration using $y\, dx$.

§ **118. Centroid.** By balancing torques about each axis we found in § 73 that the centroid of a flat plate is given by

$$\bar{x}=\frac{\int x\,dm}{\int dm}, \qquad \bar{y}=\frac{\int y\,dm}{\int dm}, \tag{10}$$

where, in the first fraction, dm is any element having a common value of x at all of its points; and in the second, a common y.

If the surface density D varies, strip elements are generally useless. But we may employ a *particle* element:

$$dm=D\,dx\,dy, \qquad \text{or} \qquad dm=D\,r\,dr\,d\theta. \tag{11}$$

All the integrals in (10) are then double integrals.

When using polar coördinates, we replace x and y in the numerator integrals of (10) by

$$x=r\cos\theta, \qquad y=r\sin\theta. \tag{12}$$

Ex. I. Determine $(\bar{x}, \bar{y})$ for a plate bounded by $y=x^2$ and $y=4$, if $D=10+x^2+y$.

The area and its density are both symmetrical with respect to the Y-axis. Clearly, then, $\bar{x}=0$.

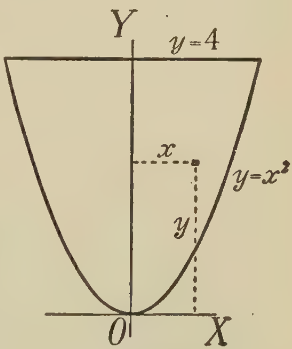


FIG. 65.

For any particle: $dm=(10+x^2+y)dy\,dx$.

$$\therefore \qquad \bar{y}=2\int_0^2\int_{x^2}^4 y(10+x^2+y)dy\,dx\div M, \tag{13}$$

where M is the same integral expression, omitting the factor y.

Study carefully the limits in (13). Also observe that we must not put $y=x^2$ or $y=4$ before performing the first integration. For these values of y are correct only at the boundaries; and we are summing up for *all* particles, — inside as well as on the edges.

Ex. II. A circular hole of diameter 2 cm. is cut tangentially through a circular plate of diameter 5 cm. (Fig. 66.) Find the centroid of the remaining portion if $D = 10 - .8\,r$, where r is measured from the point of tangency.

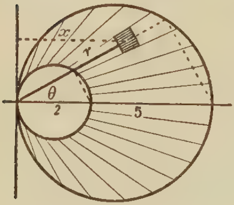


Fig. 66.

Clearly $\bar{y} = 0$. To have simple limits in finding $\bar{x}$, let us take a polar element of area:

Mass, $\qquad dm = (10 - .8\,r)\, r\, dr\, d\theta,$
Arm, $\qquad x = r\cos\theta.$

At the inner circle $r = 2\cos\theta$; at the outer, $r = 5\cos\theta$. Sectorial strips occur for every angle from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

$$\therefore \qquad \bar{x} = 2\int_0^{\frac{\pi}{2}}\int_{2\cos\theta}^{5\cos\theta}(10 - .8\,r)r^2\cos\theta\, dr\, d\theta \quad \div M, \tag{14}$$

where M is the same integral expression without the factor x.

EXERCISES

1. Find by integration the area between the circles $r = \cos\theta$ and $r = 3\cos\theta$. Check by elementary geometry. What would the limits be if we were finding the area of the larger circle alone?

2. The same as Ex. 1 for a circle of diameter 6 in. tangent internally to a circle of diameter 8 in.

3. Find the area cut off from the circle $r = 10\cos\theta$ by a vertical line 8 units from the origin.

4. The same as Ex. 3 for a line 5 units from the origin. Calculate by integration and check.

5. Find the area between the lemniscate $r^2 = 25\cos 2\theta$ and the loop of the curve $r = 3\cos 2\theta$ which it contains.

6. The same as Ex. 5 for the lemniscate $r^2 = 100\cos 2\theta$ and the rose $r = 10\cos 2\theta$. [Observe that $\sqrt{\cos 2\theta}$ is necessarily greater than $\cos 2\theta$ except at the ends.]

7. Carry out the calculation of $\bar{y}$ in (13), p. 195.

8. Calculate the same $\bar{y}$ by integrating first with respect to x, with suitable limits.

9. A triangular plate has the vertices (0, 0), (20, 0), and (0, 15). Find its total mass if $D = 10 + .04\,x + .06\,y$.

10. A plate has the shape of a quarter-ellipse with semi-axes 6 and 10. D varies thus with the distance x from the shorter side: $D = 6 - .6\,x$. Find $\bar{x}$ for the centroid.

11. The same as Ex. 10 for semi-axes 3 and 5, if $D = x + y$.

12. Find $\bar{x}$ for a quarter-circular plate of radius 20 if D varies thus with the distance r from the center of the circle: $D = 2 - .06\,r$.

13. Find the centroid of a semi-circular plate of radius a if D varies as the distance r from the center of the circle.

14. An equilateral triangular plate ABC has an altitude of 6 cm. The surface density r cm. from A is $D = 4 + r$. Find the total mass. [Take as polar axis the altitude through A.]

15. Find the centroid of the plate in Ex. 14.

16. A triangular plate ABC has $AB = 10$ cm., $\angle A = 80°$, $\angle B = 30°$. The surface density r cm. from A is $D = 2\,r$. Find the mass. [Take AB as axis; for BC, see § 12.]

17. In Ex. 16 find the distance $\bar{y}$ of the centroid from AB. [As to the integration, see Ex. 9, p. 173.]

§ 119. Moment of Inertia. We can now find the moment of inertia I for a thin flat plate whose surface density D varies with both x and y, or r and θ. The mass of any particle is

$$dm = D\,dx\,dy \qquad \text{or} \qquad dm = D\,r\,dr\,d\theta,$$

and by § 77, $dI = R^2 dm$, where R is the distance of the particle from the axis considered. Double integration will give I for the entire plate, after we have expressed R in terms of x and y, or r and θ. This can often be done by inspection of a figure.

DEFINITIONS. When I is calculated with respect to an axis in the plane of the plate, it is called a *rectangular* moment of inertia. It is denoted by I_x for the X-axis, or by I_y for the Y-axis. When calculated for an axis perpendicular to the plane, I is called a *polar* moment of inertia, — denoted by I_o if the axis passes through the origin.

Ex. I. Find I_o for a thin flat plate of constant surface density D, which has the shape of the shaded area in Fig. 67, bounded by a circle and a four-leaved rose:

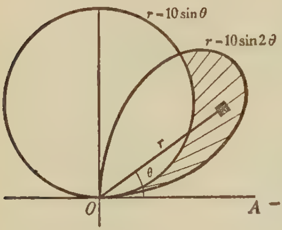


Fig. 67.

$$r=10\sin\theta, \qquad r=10\sin 2\theta. \qquad (15)$$

For any particle $D\,r\,dr\,d\theta$, the distance R from the axis (perpendicular to the plane at O) is simply r. Hence its moment of inertia is

$$dI=R^2dm=D\,r^3\,dr\,d\theta.$$

The limits for r in the first integration are given by the equations (15) of the bounding curves. The final limits for θ are found by solving simultaneously for the intersections.

Equating values of r in (15) gives $\sin 2\theta=\sin\theta$; *i.e.*,

$$2\sin\theta\cos\theta=\sin\theta$$

$$\therefore \qquad \cos\theta=\tfrac{1}{2}, \text{ or } \sin\theta=0. \qquad (16)$$

Thus $\theta=60°$ or $0°$; or, in radians, $\theta=\frac{\pi}{3}$ or 0.

Hence for the whole plate we have

$$I_0=\int_0^{\frac{\pi}{3}}\int_{10\sin\theta}^{10\sin 2\theta} D\,r^3\,dr\,d\theta=2500\,D\int_0^{\frac{\pi}{3}}(\sin^4 2\theta-\sin^4\theta)d\theta.$$

Integrating by doubling the angles twice we find

$$I_0=2500\,D\left[\frac{\sin 2\theta}{4}-\frac{5\sin 4\theta}{32}+\frac{\sin 8\theta}{64}\right]_0^{\frac{\pi}{3}}=\frac{16875\sqrt{3}}{32}D$$

$$=913.4\,D \text{ approx.}$$

§ 120. **Attraction: Electric, Magnetic, or Gravitational.** Two unlike electric charges, *concentrated at points*, attract each other with a force which varies as the product of the charges and inversely as the square of their distance apart. To find the total attraction exerted by a charge which is

distributed over a plate, upon an exterior point-charge, we resort to integration.

Similar statements hold in the case of magnetized objects, and also for gravitational attractions between masses. (Cf. *Intro.*, § 102.)

Ex. I. A circular plate of radius 10 cm. carries a uniform electric charge of 2 units per sq. cm. Find the total attraction upon an exterior point-charge of 4 units located 20 cm. away from the plate on the perpendicular axis. (Fig. 68.)

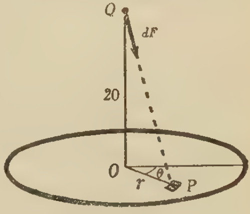


FIG. 68.

The charge on an element of area $r\,dr\,d\theta$ is $2\,r\,dr\,d\theta$. Its attraction upon the charge Q is

$$dF = \frac{k(4)(2\,r\,dr\,d\theta)}{\overline{QP}^2}, \tag{17}$$

where k is a constant.

This attraction acts along the line QP. Other elementary attractions will act along other directions; and it would be incorrect to add them together numerically to get their total effect. It is clear from symmetry, however, that the resultant attraction will be toward the center of the plate. Hence we consider merely the component of the elementary attraction dF in the direction QO; that is, dF times $\cos(\angle PQO)$.

The part of the resultant force contributed by dF is then

$$dR = dF\cdot\cos(\angle\, PQO) = \frac{8\,k\,r\,dr\,d\theta}{\overline{QP}^2}\cdot\frac{20}{\overline{QP}} = \frac{160\,k\,r\,dr\,d\theta}{\overline{QP}^3}. \tag{18}$$

Now $\overline{QP} = \sqrt{20^2+r^2}$, whence

$$\overline{QP}^3 = (400+r^2)^{\frac{3}{2}}.$$

Therefore, the resultant or total attraction is

$$R = \int_0^{2\pi}\int_0^{10}\frac{160\,kr\,dr\,d\theta}{(400+r^2)^{\frac{3}{2}}} = 160\,k\int_0^{2\pi}\left[-\frac{1}{\sqrt{400+r^2}}\right]_0^{10}d\theta$$

$$= k\pi\left[16-\frac{32}{\sqrt{5}}\right].$$

Remark. If the charge Q were not located symmetrically with respect to the charge on the plate, we should have to find not only the component perpendicular to the plate, but also components parallel to the plate in two directions. The three components could then be combined to find the resultant attraction.

EXERCISES

In Ex. 1–8, find the moment of inertia, with respect to the axis mentioned, of a flat plate having the specified shape and surface density D.

1. Quarter-circular plate of radius 10 cm., with $D = 6 - .6\,r$, where r is the distance from the center O of the circle:

(*a*) Axis, one of the straight sides of the plate,

(*b*) Axis perpendicular to the plate at O.

2. Like Ex. 1 if the radius is 20 cm. and $D = 2 - .06\,r$.

3. Circular plate, radius 10 cm., $D = k$: axis perpendicular to the plate at a point O on the circumference.

4. Like Ex. 3, if D varies inversely as the distance r from O, and $D = 2$ when $r = 1$.

5. Like Ex. 3, if the radius is a, and if D varies thus with the distance r from O: $D = 10 + .06\,r$.

6. Rectangular plate, 5 cm. by 10 cm., about the longer side as axis, if D varies thus with the distances, x cm. and y cm., from the shorter and longer sides: $D = 60 - x - y$. Also find the radius of gyration K.

7. Right triangular plate with sides 4 cm. and 8 cm., about the shorter side, if D varies as in Ex. 6. Also find K.

8. A plate shaped like one loop of the four-leaved rose $r = 10 \cos 2\theta$, with $D = k$, about an axis OA perpendicular to the plate at the origin O.

9. If the plate in Ex. 8 be held in a vertical plane, with its axis of symmetry inclined at an angle B from the vertical, find the torque of its weight with respect to the (then horizontal) axis OA.

10. Find the centroids of the plates mentioned above:

(*a*) In Ex. 4; (*b*) In Ex. 5.

In Ex. 11–14, find the resultant force exerted upon a particle or concentrated electric charge, of mass or strength m, by an attracting bar, plate, or distributed charge, as specified. The density D is constant in each case.

11. Circular plate of radius 5 cm.; particle 12 cm. away, on the line perpendicular to the plate at its center.

12. Rod 8 cm. long; particle 3 cm. away from the nearest point:

(*a*) In line with the rod;

(*b*) On the perpendicular to the rod at the center.

13. (*a*), (*b*). Like Ex. 12 (*a*), (*b*), for a rod L cm. long and a particle H cm. away.

14. Uniform charge on a plate 20 cm. square; point charge 5 cm. away on the perpendicular to the plate at its middle. [Hint: Consider a quarter of the plate, in the shape of a right triangle.]

15. A particle P is in line with a rod 50 cm. long X cm. away. Find the work done by the attraction F in moving P from $X = 20$ to $X = 10$. [Hint: First find F for any distance X.]

§ 121. Theorems of Pappus. Two very useful theorems concerning centers of gravity were discovered by Pappus of Alexandria about 300 A.D.:

(I) If a plane area be revolved about an axis outside the area but lying in its plane, *the volume of revolution so generated is equal to the area revolved times the distance traveled by the centroid of that area.*

(II) If a plane curve be revolved about an axis in the plane, which does not cut the curve, *the surface of revolution so generated is equal to the length of arc revolved times the distance traveled by the centroid of that arc.*

Both theorems are true either for a complete revolution or for a partial revolution. (They will be proved in § 122.)

By means of these theorems, if we know any two of the three quantities in either group below, we can find the third:

Area revolved,	Length of arc revolved,
Volume generated,	Surface generated,
$\bar{y}$ for centroid of area;	$\bar{y}$ for centroid of arc.

Ex. I. Find the centroid for a uniform semi-circular rod of radius 10 in. (Fig. 69.)

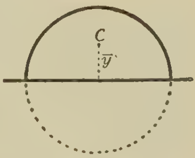


Fig. 69.

Revolution about an axis through the ends of the rod would generate a spherical surface.

$$\text{Arc revolved,}\quad \tfrac{1}{2}(2\,\pi r) = 10\,\pi.$$
$$\text{Area generated,}\quad 4\,\pi r^2 = 400\,\pi.$$

The distance traveled by C is $2\pi\bar{y}$. Hence by Theorem II:

$$400\pi = (10\pi)(2\pi\bar{y}), \tag{19}$$

$$\bar{y} = \frac{20}{\pi} = 6.37, \text{ approx.}$$

Ex. II. Find the area of the convex outer surface of a ring generated by revolving a semi-circle of radius 10 in. about an axis 15 in. from the flat side. (Fig. 70.)

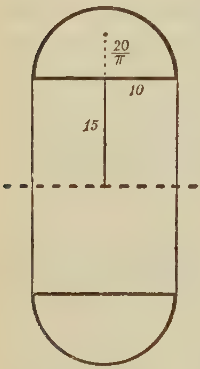


Fig. 70.

The outer surface is generated by an arc whose length is 10π in., and whose centroid by Ex. I is $\left(15+\frac{20}{\pi}\right)$ in. from the new axis of rotation. Hence, by Theorem II again, the surface area (S sq. in.) is

$$S = (10\pi)\left[2\pi\left(15+\frac{20}{\pi}\right)\right]$$
$$= 300\pi^2 + 400\pi. \tag{20}$$

§ **122. Proof of Theorems.** The first theorem of Pappus can be proved as follows.

The volume generated by an element dA, or $dy\,dx$, at a distance y from the axis of revolution is $2\pi y\,dy\,dx$, and the entire volume is

$$V = 2\pi\int_a^b\int_{y_1}^{y_2} y\,dy\,dx, \tag{21}$$

where y_1 and y_2 are the lower and upper ordinates of the bounding curve at any x; and a, b are the extremes of x.

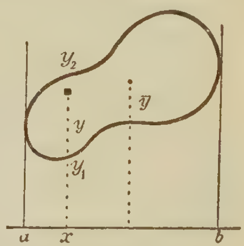


Fig. 71.

Also, the ordinate $\bar{y}$ of the centroid of area is

$$\bar{y} = \int_a^b\int_{y_1}^{y_2} y\,dA \div A. \tag{22}$$

Thus $V = (2\pi\bar{y})$ times A, as stated by Theorem I. (Q.E.D.)

The proof of Theorem II runs likewise, but with two integrals each time, corresponding to the upper and lower bounding arcs. That is,

$$S=\int_a^b 2\,\pi y_2\sqrt{1+\left(\frac{dy_2}{dx}\right)^2}dx+\int_a^b 2\,\pi y_1\sqrt{1+\left(\frac{dy_1}{dx}\right)^2}dx, \qquad (23)$$

while $\bar{y}$ for the curve equals this same sum, without the factor $2\,\pi$, and divided also by the length s. Thus

$$S=(2\,\pi\bar{y})\ (s). \qquad \text{(Q.E.D.)}$$

Note. For a partial revolution the factor $2\,\pi$ would everywhere be replaced by the number of radians through which the figures are rotated. Thus the relation between the distance traveled by the centroid and the volume or area generated still stands, as stated in § 121.

EXERCISES

Each axis of revolution lies in a plane with the given arc or area.

1. Find the volume of a flat ring or washer generated by revolving a rectangle, with sides 6 in. and 1 in., about an exterior axis 4 in. from a short side. Check by geometry.

2. Like Ex. 1 for an axis 4 in. from a long side. What shape has the solid this time?

3. Find the volume and surface area generated by revolving any circle about a tangent line.

4. Find the volume and area of an automobile tire 32 in. high, if the cross section of the tire itself is a circle of diameter 4 in.

5. Like Ex. 4 for a tire 40 in. by 6 in.

6. An elliptical anchor ring is generated by revolving an ellipse, of semi-axes 5 in. and 3 in., about a line L parallel to the minor axis and 8 in. from the center. Find its volume.

7. Like Ex. 6, if L is parallel to the major axis.

8. Find the centroid of area for any semi-circle. Hence what centroid for a quarter-circular area?

9. Find the centroid of a quarter-circular arc. Hence what centroid for a semi-circular arc?

10. A bulge at the base of a half-cylindrical column has the shape generated by revolving a quarter circle of radius 6 in. about a line 10 in. from one straight side, through 180°. Find the volume and area of the bulge. [The results of Ex. 8 and 9 may be used.]

11. Find the centroid of a right triangle with legs 9 ft. and 30 ft.

12. A cut running 120° around a hill has the shape generated by revolving the triangle in Ex. 11 about an axis 100 ft. from the short leg. Find the volume of earth removed.

13. Prove that for any flat plate $I_0 = I_x + I_y$. (§ 119.)

14. Prove that the moments of inertia of a flat plate with respect to an axis through the centroid (say I_g) and any parallel axis k units distant (say I_p) are related thus: $I_p = I_g + k^2M$, where M is the mass. [Hint: Taking the axis through the centroid as the X-axis: $I_g = \int\int y^2 D dy dx$, while $I_p = \int\int (y+k)^2 D dy dx$. Expand the latter and show that the middle term must be zero since $\bar{y} = 0$.]

15. Find the moment of inertia of a uniform circular plate of radius 10 cm. with respect to a diameter. Then, by using the theorem in Ex. 14, find I also with respect to lines as follows:

(a) 3 cm. from the center, (b) tangent, (c) 20 cm. from the center.

16. Like Ex. 15 for a rectangular plate 8 cm. by 6 cm., taking the central axis parallel to the longer side.

PART II. COÖRDINATES IN SPACE

To study solids and curved surfaces effectively some further ideas of coördinates are needed.

§ **123. Points in Space.** The location of a point P in space is usually described by giving its distances x, y, z, from three mutually perpendicular reference planes, such as YOZ, ZOX, XOY, as in Fig. 72.

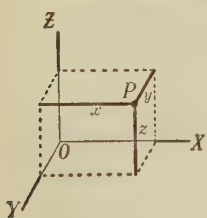


Fig. 72.

Virtually this idea was used in § 44 when we plotted points on a surface by calculating the height z above any specified point (x, y) of a chosen base plane.

Each of the "rectangular coördinates," x, y, z, is considered positive in one direction and negative in the opposite. It is customary to draw the positive direction of OX to the right, of OY toward the observer, and of OZ upward. (The X- and Y-axes would

then appear in their usual relation if the plane XOY were viewed from below, *i.e.*, looking in the positive z-direction.)

The three reference planes separate all space into eight parts called octants. That one in which x, y, z are all positive is called "the first octant."

Cylindrical Coördinates. Another useful way to describe a point P is to use polar coördinates, r, θ, in a base plane, together with the elevation z above or below that plane. (Fig. 73.)

More adequately stated: Let z be the positive or negative distance of P from a basic reference plane QOM. Let r be the positive distance of P from an axis OZ which is perpendicular to QOM. And let θ be the angle which the plane OQP through OZ and P makes with a fixed plane OMN through OZ.

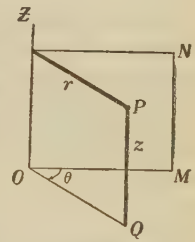


FIG. 73.

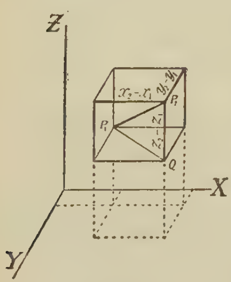


FIG. 74.

The values z, r, θ are called the "cylindrical coördinates" of P.

The relations between rectangular and cylindrical coördinates are simple:

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$r = \sqrt{x^2 + y^2}, \tag{24}$$

and z is common.

§ 124. **Distance Formulas.** The distance between any two points P_1 and P_2 in space is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{25}$$

For, planes through P_1 and P_2 parallel to the three coordinate planes inclose a rectangular space with diagonal P_1P_2, and with edges $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$. The horizontal

projection of P_1P_2, viz. P_1Q, equals $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. And $\overline{P_1P_2}^2=\overline{P_1Q}^2+\overline{QP_2}^2$, which gives (25).

The corresponding formula in cylindrical coördinates is rarely needed, but can be found by replacing x_2 by $r_2 \cos \theta_2$, x_1 by $r_1 \cos \theta_1$, etc.

Distance from an Axis. The distance of any point P from the Z-axis is simply r, or $\sqrt{x^2+y^2}$. Similarly, as may easily be shown, the distance of P from the X-axis is $\sqrt{y^2+z^2}$; and from the Y-axis, $\sqrt{x^2+z^2}$.

§ **125. Equation and Locus.** The total set of points in space whose coördinates satisfy a given equation is called the locus of the equation. There may be no real locus.

A single equation involving x, y, and z, ordinarily represents a *surface*. For it determines z as some function of x and y, say

$$z=f(x, y); \tag{26}$$

and, as x and y vary, (26) usually defines a moving point, which lies always along a surface, — one which can be plotted from (26) as in § 44.

It may happen for some equations, however, that the locus is real only at isolated points or along a curve.

Even when one or two coördinates are missing, a single equation represents some surface. *E.g.*, the equation $z=5$ is true for all points in a horizontal plane five units above the plane XOY, and nowhere else. Again, $x^2+y^2=25$ is satisfied not only at any point P on a certain circle of the plane XOY; but also at any point Q in space, which is directly above or below such a point P, — in other words, at all points on a *cylinder* of radius 5 whose axis is the Z-axis, and nowhere else.

The curve of intersection of two surfaces is represented by a *pair of equations*, viz. the equations of the two surfaces taken simultaneously, — not by any single equation obtained

by combining the two. (Such a combined equation would represent *some other surface* passing through the required intersection, — just as a combination of the equations of two plane curves represents another curve through their points of intersection. Cf. § 1.)

In Chapter I we regarded a single equation in x and y as representing a curve rather than a surface. But we were then working exclusively in one plane, say $z=0$. To represent such a curve now, when dealing with three-dimensional space, we need two equations, — the former equation and $z=0$.

§ **126. Drawing a Surface by Sections.** In drawing a curved surface, — whether a sphere or cone, or something more complicated, — what we really do is to draw in perspective a number of sections of the surface, or lines on the surface, which show its character.

Such sections can be determined from the equation of a surface by calculating points, as in § 44 and *Intro.*, § 296. But for some surfaces we can recognize by inspection the nature of the sections made by certain planes, notably the coördinate planes and other planes parallel to them.

Ex. I. Draw the surface $\frac{x^2}{25}+\frac{y^2}{9}-\frac{z^2}{16}=1.$

The section in the plane $z=0$ is a curve in which

$$\frac{x^2}{25}+\frac{y^2}{9}=1. \tag{27}$$

Evidently this is an ellipse with semi-axes 5 and 3.

Hence we first draw $ABCD$ in Fig. 75, with the axes in perspective.

The other two coördinate planes form sections as follows: $y=0$, an hyperbola with transverse axis along the X-axis.

This hyperbola must meet the above ellipse at A and C, points on the surface at which both y and z are zero.

$x=0$, an hyperbola with transverse axis along the Y-axis.

This hyperbola must pass through B and D.

For any other horizontal plane, $z=k$, we write:

$$\frac{x^2}{25}+\frac{y^2}{9}=1+\frac{k^2}{16},$$

or

$$\frac{x^2}{25\left(1+\frac{k^2}{16}\right)}+\frac{y^2}{9\left(1+\frac{k^2}{16}\right)}=1. \tag{28}$$

This equation, taken alone, holds for all points on a cylinder with an elliptic base, but those points in the given plane $z=k$ form an ellipse having

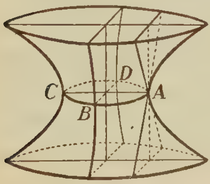


FIG. 75.

$$a=5\sqrt{1+\frac{k^2}{16}}, \qquad b=3\sqrt{1+\frac{k^2}{16}}. \tag{29}$$

These semi-axes steadily increase with k.

The above sections suffice for drawing a portion of the surface as in Fig. 75. Properly, the surface continues infinitely far up and down, and hence outward. Of course, it is open or hollow in the middle.

If we desire further information, we may show that most of the sections made by planes $x=k$ or $y=k$ are hyperbolas, the direction of whose axes depends upon k. But a few of those planes form sections which are straight lines. *E.g.*, the plane $x=5$ gives two lines, along which

$$\frac{y^2}{9}=\frac{z^2}{16}, \qquad \text{or} \qquad z=\pm\frac{4}{3}y.$$

To draw these two lines, we locate some points where the surface meets the plane $x=5$. This can be done by drawing lines $x=5$ in any horizontal elliptic section.

EXERCISES

1. Where are all points in space for which $z=0$? All for which $y=6$? $r=5$? $\theta=\frac{\pi}{3}$? $y=x$? $z=x$? $z=r$?

2. Draw the three coördinate axes in perspective. Plot the points $A(4, 2, 0)$, $B(4, 2, 5)$, $C(4, 2, -5)$, $D(0, 0, 8)$, $E(0, 6, 0)$, $F(-6, -4, 5)$. Also calculate the distances AD, BE, BF, EF.

3. Find the distances of the following points from the Z-axis: $G(9, 12, 16)$, $H(4, -3, 0)$, $K(6, 0, -8)$, $L(0, -5, -12)$. Likewise from the Y-axis, and from the X-axis.

4. The electric potential at a point P due to a charge m at a point Q is $m/\overline{PQ}$. Calculate this if $m=65$, and if P and Q are respectively $(0, 0, 0)$ and $(3, 4, 12)$. If $(9, -2, -5)$ and $(-3, 7, 15)$. If $(7, 6, 8)$ and $(7.01, 6.02, 8.01)$.

5. Find the cylindrical coördinates of $(3, 3, 7)$. Of $(0, 5, 8)$. Of $(-2, -2, -6)$. Of $(-4, 0, 0)$.

6. Find the rectangular coördinates of points (r, θ, z) as follows:

$$\left(10, \frac{3\pi}{2}, -8\right);\ \left(10, \frac{\pi}{6}, 3\right);\ \left(\sqrt{8}, \frac{5\pi}{4}, 0\right);\ (20, 20, -20).$$

7. Draw a sphere. Notice what curves you use to bring out its three-dimensional character. Similarly for a cylinder and for a cone.

8. Draw the following surfaces by considering sections made by the planes $x=0$, $y=0$, $z=0$; $z=\pm 2$, ± 4.

(a) $x^2+y^2+z^2=16$, (b) $x^2+y^2=z$.

Also transform equations (a), (b) into cylindrical coördinates.

9. What sort of curve is the section of $4x^2+9y^2+25z^2=3600$, which is made by the plane $x=0$? By $y=0$? $z=0$? $z=\pm 10$? $y=20$?

10. The same as Ex. 9 for these surfaces:

(a) $x^2+y^2=10z$, (b) $x^2-y^2=10z$,
(c) $x^2+y=10z$, (d) $x+y=z$.

11. What is the locus of each of the following equations in the XY-plane alone? In space?

(a) $x^2+y^2=100$, (b) $(x-4)^2+y^2=25$,
(c) $x^2+y^2=6y$, (d) $2x+3y=12$,
(e) $y=2x$, (f) $x=6$,
(g) $x^2+y^2+z^2=0$, (h) $(x^2-y)^2+z^2=0$.

12. Plot the portion of the surface $z=x^2+y^2$ inclosed by the four planes $x=0$, $x=4$, $y=0$, $y=2$. (Cf. § 44.)

13. Eliminate z from the two equations $z=x$, $x^2+2y^2+z^2=50$. What kind of locus in space does the resulting equation have? What relation must it have to the two given surfaces?

14. The same as Ex. 13 for $x^2+z^2=100$ and $y^2+z^2=100$.

§ 127. The Conicoids. For reasons which will appear later, the locus of any quadratic equation in x, y, and z, if

real, is called a *conicoid.* With minor exceptions, every plane section is an ellipse, parabola, or hyperbola.

The chief types of conicoids have the equations on p. 211. They can be drawn by the methods of § 126, and are found to have the forms shown in Fig. 76. It is not necessary to memorize these types, but we should be able to recognize them quickly by considering sections. Study carefully the statements made concerning the various sections shown.

It will be observed that each of these surfaces is symmetrical with respect to at least two of the coördinate planes, since changing the sign of x or y (and in some cases z) does not affect the equation.

Also it will be seen that the volume within any of the first four surfaces, between suitable planes, is readily found by a single integration. For, the cross sections in at least one direction are ellipses; and the area "πab" is expressible in terms of one variable and the given constants a, b, etc.

Ex. I. Find the volume within the surface

$$\frac{x^2}{25}+\frac{y^2}{9}-\frac{z^2}{16}=1,$$

from $z=0$ to $z=10$.

By (29), p. 208, every horizontal section, at any elevation z, is an ellipse whose semi-axes are:

$$a=5\sqrt{1+\frac{z^2}{16}}, \qquad b=3\sqrt{1+\frac{z^2}{16}}.$$

Hence the area of the section is

$$A_s=\pi ab=15\,\pi\left(1+\frac{z^2}{16}\right). \tag{30}$$

Integrating with respect to the distance z:

$$V=\int_0^{10} A_s\,dz=15\,\pi\left[z+\frac{z^3}{48}\right]_0^{10}=462.5\,\pi. \tag{31}$$

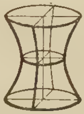
IMPORTANT TYPES OF CONICOIDS

Name **Equation** **Form**

Ellipsoid: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Ellipsoid

The principal sections, $x=0$, $y=0$, $z=0$, are all ellipses; likewise $x=k$, $y=k$, $z=k$, until the latter sections reduce to points and then become imaginary.

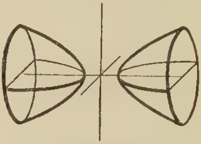


Hyperboloid of 1 Sheet

Hyperboloid of 1 Sheet: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

Two principal sections, hyperbolas; one, an ellipse. In general, $x=k$, $y=k$, hyperbolas; $z=k$, ellipse.

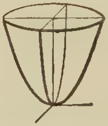
Hyperboloid of 2 Sheets: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$



Hyperboloid of 2 Sheets

Two principal sections, hyperbolas; one, imaginary. Sections $x=k$ are real if $k^2>a^2$, and are ellipses.

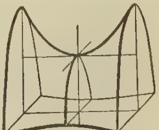
Elliptic Paraboloid: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z.$$



Elliptic Paraboloid

Sections $x=0$, $y=0$, parabolas with an upward axis. Section $z=0$, a point. In general, $z=k$ (positive), an ellipse; $z=k$ (negative), imaginary.

Hyperbolic Paraboloid: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z.$$



Hyperbolic Paraboloid

FIG. 76.

Sections $x=0$ or k, parabola with axis downward; $y=0$ or k, parabola with axis upward. Section $z=0$, two straight lines; $z=k$ (positive), hyperbola with axis parallel to X-axis; $z=k$ (negative), hyperbola with axis parallel to Y-axis. This surface is often called a "saddle surface."

Remark. Ellipsoids and other conicoids, being merely *surfaces*, do not have volumes. But they may *inclose* volumes, in whole or in part.

Such an inclosed volume we shall often refer to briefly as the volume of the conicoid in question.

§ 128. Contours and Profiles. A horizontal section of any surface is called a *contour;* a vertical section, a *profile.* We have used both, drawn in perspective, to show the shape of various surfaces.

Sometimes perspective drawing is not satisfactory and the surface is better displayed by contours alone, all projected down upon one plane as if seen from directly above.

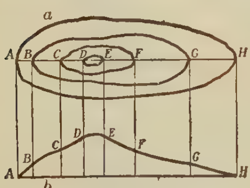


Fig. 77.

This is what is done in topographic maps. *E.g.*, Fig. 77 *a* represents a hill, the contours having a difference of elevation of 10 ft. Where these are crowded the hill is steep.

From such a map we can draw the profile of the hill in any direction, say along the line AH. Simply lay off on a parallel line the horizontal distances between the contours shown, and erect ordinates showing the elevation belonging to each contour. (Fig. 77 *b*.)

To illustrate the plotting of a mathematical surface by contours only, consider the equation

$$xy = z. \tag{32}$$

(Profiles formed by planes $x=k$, $y=k$ are straight lines in parallel planes but with different slopes, and are difficult to draw effectively with the axes in their usual position. Try it, if you have time.)

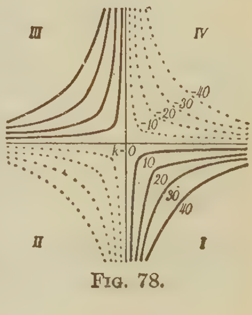


Fig. 78.

The contour $z=0$ consists of the Y- and X-axes, $x=0$ and $y=0$, respectively. Every other contour, $z=k$, is a rectangular hyperbola, in Quadrants I and III when k is positive, and in Quadrants II and IV when k is negative. These are shown in Fig.

78, as the quadrants would appear from *above*. The surface rises from the plane $z=0$ in I and III, and falls away in II and IV. It is a saddle surface or hyperbolic paraboloid, somewhat like that in Fig. 76, but rotated horizontally through 45°.

EXERCISES

1. Determine any helpful sections and draw the following surfaces. Compare your drawings with Fig. 76.

(*a*) $\frac{x^2}{16}+\frac{y^2}{4}+\frac{z^2}{9}=1,$ (*b*) $\frac{x^2}{16}+\frac{y^2}{4}-\frac{z^2}{9}=1,$

(*c*) $\frac{x^2}{16}+\frac{y^2}{4}=z,$ (*d*) $\frac{x^2}{16}-\frac{y^2}{4}-\frac{z^2}{9}=1,$

(*e*) $\frac{x^2}{16}-\frac{y^2}{4}=z,$ (*f*) $\frac{x^2}{16}+\frac{y^2}{4}=z^2.$

2. Draw on a large scale the contours of the surface in Ex. 1 (*c*) at the levels $z=4, 3, 2, 1, 0$. By measurement as in Fig. 77 construct the profile of the surface in the plane $y=0$. Check by plotting that profile directly from the equation in Ex. 1 (*c*).

3. The same as Ex. 2 for the surface in Ex. 1 (*e*).

4. Draw roughly on a large scale a figure representing contours of a fairly regular hill at elevations 0, 10 ft., 20 ft., etc., above the base. Show the shape of the profile in some plane AB.

5. In the following figure, (*a*) shows the profile of a rock on a sea-coast. The shore is steep, but fairly straight; the rock is nearly round at the water line. Draw contours to represent these facts roughly.

6. In the figure above, (*b*) shows contours of some foothills between Mt. Hood and the Columbia River. Draw the approximate profiles in the directions CC' and DD'.

7. What is the area of the section of the surface in Ex. 1 (*c*) made by the plane $z=9$? $z=h$? Find the volume within that surface from $z=0$ to $z=6$.

8. Like Ex. 7 for the surface in Ex. 1 (*b*).

9. Find the volume of the ellipsoid in Ex. 1 (*a*).

10. Find the distance from the point (1, 2, 4) to the point on the surface in Ex. 1 (*e*) at which $x=8$ and $y=-6$.

11. What is the locus of $x^2+y^2=49$ in space? Of $y=2x$?

§ 129. Type Equations of the Elementary Surfaces. In finding the areas of curved surfaces, and in some other problems, we shall need to know the equations of various familiar surfaces, in certain positions.

(*A*) *Sphere.* Let a sphere of radius a have the center (g, h, k). Then for any point (x, y, z) on the surface we have by the distance formula:

$$(x-g)^2+(y-h)^2+(z-k)^2=a^2. \tag{33}$$

And this equation is true only for points on the sphere.

If the center is at the origin, (33) reduces to

$$x^2+y^2+z^2=a^2; \tag{34}$$

or $r^2+z^2=a^2$ in cylindrical coördinates.

(*B*) *Cylinder.* Let a circular cylinder of radius a have the Z-axis for its own axis. Then for every point (x, y, z) of the surface, — regardless of the value of z:

$$x^2+y^2=a^2. \tag{35}$$

And (35) holds true only at points on the cylinder.

We use the term "cylinder" here to designate not a solid but a cylindrical surface, *i.e.*, a surface generated by an unlimited straight line moving parallel to a fixed straight line and always intersecting a fixed curve, — here a circle. Similarly, by a "cone" we mean a surface generated by an unlimited straight line moving so as to pass through a fixed point and always intersect a fixed curve.

When we wish to consider a limited portion of either surface, or the space or volume inclosed thereby, up to a certain distance, we shall speak

of a cylinder or cone of a certain height, or shall otherwise indicate what is meant.

For either a cylinder or cone, any position of the moving line or generatrix (*i.e.*, any straight line on the surface) is called an "element."

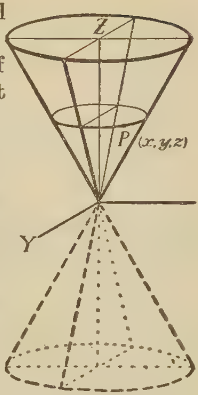


FIG. 79.

(*C*) *Cone.* Let a cone have its vertex at $(0, 0, 0)$ and its axis in the Z-axis. Let some horizontal section, — and hence every horizontal section, — be an ellipse, or a circle as a special case. Then at any point $P(x, y, z)$ of the surface:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where a and b are some lengths proportional to z. *I.e.*, $a=pz$ and $b=qz$, where p and q are the numerical values of the semi-axes in the section $z=1$. Then

$$\frac{x^2}{p^2}+\frac{y^2}{q^2}=z^2 \tag{36}$$

for all points on the cone and for no others.

(*D*) *Plane.* The equation of every plane is linear in x, y, and z, of the general form:

$$lx+my+nz+k=0. \tag{37}$$

Proof. (I) If the plane is *vertical*, it must cut the plane XOY in some straight line

$$lx+my+k=0; \tag{38}$$

and this equation is true for every point (x, y, z) in the plane, regardless of z; and for no other points.

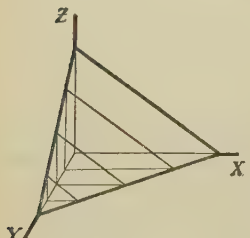


FIG. 80.

(II) If the plane is *non-vertical*, every section perpendicular to the Y-axis has some constant slope l, which is the same for all such sections: *i.e.*,

$$\frac{\partial z}{\partial x}=l.$$

Hence an equation expressing z in terms of x (and possibly y) can involve x only in one term lx. If it includes terms involving y, call those terms $F(y)$, and we may write

$$z=lx+F(y)+k. \tag{39}$$

Further, considering sections perpendicular to the X-axis, we have $\partial z/\partial y=m$, some constant. Hence $F(y)$ is simply my.

$$\therefore \qquad z=lx+my+k. \tag{40}$$

This is of the first degree, and is included under (37) when $n=-1$, just as (38) is when $n=0$. Thus for any plane the equation has the form (37).

If a plane passes through three given points, we can get its equation in a definite form by substituting the coördinates of those points in (37), solving for three of the constants l, m, n, and k, substituting back, and dividing out the remaining constant.

In particular, if a plane does not pass through the origin but cuts off intercepts a, b, c, on the axes, we substitute $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ in (37) and get

$$la+k=0, \qquad mb+k=0, \qquad nc+k=0.$$

Hence $l=-k/a$, $m=-k/b$, $n=-k/c$; and (37) becomes

$$-\frac{k}{a}x-\frac{k}{b}y-\frac{k}{c}z+k=0.$$

Or, on dividing out k (which is not zero):

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \tag{41}$$

By this equation we can write at sight the equation of any plane whose intercepts are known.

§ **130. Locating Intersections Graphically.** In calculating a volume bounded by two different surfaces it is important to see where the surfaces meet and to be able to draw the intersection. A good method is to cut the two surfaces by some plane whose intersections with both can be recognized,

and see where these cross. Then cut by another such plane, and another, until enough common points are known.

For the elementary surfaces convenient planes are easily chosen. Sections made by planes through the axis or through an element of a cylinder or cone, or through the vertex of a cone, are especially simple. Any plane section of a sphere is a circle.

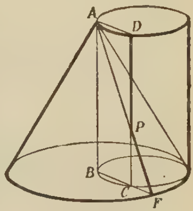


FIG. 81 *a*.

Ex. I. A cone of radius 10 and height 18 is cut by a cylinder of diameter 10, one of whose elements coincides with the axis of the cone. Draw their intersection.

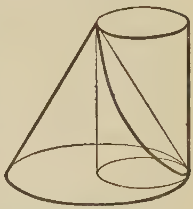


FIG. 81 *b*.

Any plane through the axis AB cuts the cone in some element AF and the cylinder in some element CD. The point P where these meet is common to the cone and cylinder (Fig. 81 *a*). Other plane sections give other positions of P; and the required intersection, viewed in this perspective, appears as in Fig. 81 *b*.

EXERCISES

1. Write the equation in rectangular coördinates of a sphere with
(*a*) Center (0, 0, 0), radius 9; (*b*) Center (8, -4, 5), radius 12;
(*c*) Ends of a diameter at (0, 0, 0) and (0, 0, 10).

2. In Ex. 1 (*a*) find also the equation in cylindrical coördinates.

3. Write the equation in rectangular coördinates of a plane

(*a*) Making intercepts of 7, 2, and 9 on the X-, Y-, and Z-axes;

(*b*) Cutting the Z-axis at (0, 0, -6) and the plane XOY in the line $x-2y=8$;

(*c*) Cutting the planes XOZ and YOZ in the lines $2x+3z=18$ and $3y+2z=12$;

(*d*) Passing through the points (2, 1, 3), (4, 0, 0), and (0, 0, 8);

(*e*) Parallel to the Z-axis, with X- and Y- intercepts of 3 and 2;

(*f*) Parallel to the X-axis, with Y- and Z- intercepts of 2 and -5;

(*g*) Bisecting any angle between the planes XOY and YOZ.

4. Find the equations in rectangular coördinates and in cylindrical coördinates for the following cylinders:

(*a*) With radius 6 and axis in the Z-axis; in the Y-axis; in the X-axis;

(*b*) Axis along OZ, cross section elliptical with $a=5$, $b=3$;

(*c*) Axis parallel to OZ through (10, 0, 0), radius 10.

5. Like Ex. 4 for cones with axes in OZ and sections as follows:

(*a*) Circle of radius 4 in the plane $z=1$;

(*b*) Circle of radius 1 in the plane $z=10$;

(*c*) Ellipse with semi-axes 10 and 8 in the plane $z=2$;

(*d*) Circle in any plane $z=k$; vertex angle of cone, 90°;

(*e*) Like (*d*) with a vertex angle of 110°.

6. What kind of locus has each following equation?

(*a*) $x^2+y^2+z^2=12$, (*b*) $x^2+(y-3)^2+(z+6)^2=1$, (*c*) $r^2+z^2=36$,
(*d*) $x^2+y^2+z^2=4z$, (*e*) $x^2+y^2=70$, (*f*) $x^2+4y^2=36$,
(*g*) $x^2+z^2=49$, (*h*) $x^2+4y^2=z^2$, (*i*) $x+y=8$,
(*j*) $x^2+4y=0$, (*k*) $r=\frac{1}{2}z$, (*l*) $r=10\cos\theta$.

7. Draw each of the following elementary surfaces, and show several plane sections as specified:

(*a*) Cylinder, with radial planes through the axis;

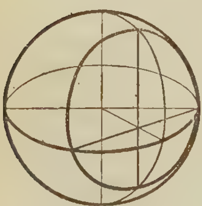
(*b*) Cylinder, with planes through an element;

(*c*) Cone, with planes through the axis;

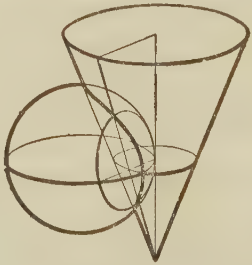
(*d*) Cone, with non-axial planes through the vertex;

(*e*) Sphere, with planes through the vertical diameter.

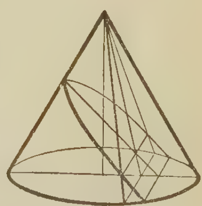
8. (*a*)–(*c*). Redraw the following figures (*a*)–(*c*) on a larger scale, determining several points of the intersections by means of approximate plane sections taken as illustrated. [Cf. Ex. 7.]



(*a*)
Sphere with non-central plane.



(*b*)
Sphere with cone; axis tangent.



(*c*)
Cone with any plane.

9. Redraw Fig. 81 with the cone inverted.

10. Draw, showing the intersection in each case:

(*a*) A cylinder cut by an inverted cone, of like height and radius, whose axis is an element of the cylinder;

(*b*) A sphere cut by a cylinder, whose diameter equals the radius of the sphere, and one of whose elements passes through the center;

(*c*) A sphere cut by an inverted cone whose vertex lies on the sphere and whose axis passes through the center;

(*d*) Like (*c*) but with the axis of the cone tangent to the sphere.

11. The ellipsoid $9x^2+25y^2+16z^2=3600$ is cut by the cylinder $x^2+4y^2=144$. Draw the part of the figure in the principal octant. Show sections made by several planes, $x=k$.

12. Like Ex. 11 for the paraboloid $x^2+4y^2=20z$ and the plane $x+y+z=10$. [Draw the paraboloid up to $z=20$.]

§ 131. Volumes by Double Integration. Our basic method of finding a volume is by a single integration,

$$V=\int A_s\,dx.$$

But this presupposes that we know a formula for the area A_s of a plane cross section in terms of the perpendicular distance x from some fixed point.

When we do not know such a formula, we can often find it by a preliminary integration. Thus, as was explained in *Intro.*, § 297, the volume is obtained by double integration:

$$V=\int\int z\,dy\,dx \tag{42}$$

Let us now state the matter in another way. If we do not know the volume $A_s\,dx$ of a slice of tiny thickness dx, we may regard the slice as composed of still smaller elements, viz. narrow columns. By summing these columns we shall get the slice; and by summing slices get the entire volume.

Let us illustrate in a problem where single integration is possible as a check.

Ex. I. Find the volume of the ellipsoid

$$x^2+4y^2+9z^2=36. \tag{43}$$

By symmetry, one-eighth lies in the first octant where x, y, and z are all positive. Drawing principal sections as in § 127, but reducing the scale for x, gives the shape shown in Fig. 82. *E.g.*, the base is bounded by a quarter of the ellipse:

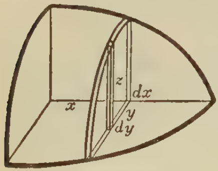


FIG. 82.

$$x^2+4y^2=36, \qquad z=0. \tag{44}$$

Clearly the surface does not overhang, or protrude beyond, this base.

Any slender column-element with tiny base $dx\,dy$ has some height z and volume $z\,dx\,dy$. By (43) this is

$$z\,dx\,dy=\tfrac{1}{3}\sqrt{36-x^2-4y^2}\,dx\,dy.$$

Summing for all possible columns will give $\frac{1}{8}V$. To do this, we simply run $dx\,dy$ or $dy\,dx$ over the entire base (quarter-ellipse). Integrating first with respect to y the limits will be 0 and $\frac{1}{2}\sqrt{36-x^2}$. [See (44).] Then, with respect to x, the limits will be 0 and 6:

$$\tfrac{1}{8}V=\int_0^6\int_0^{\frac{1}{2}\sqrt{36-x^2}}\tfrac{1}{3}\sqrt{36-x^2-4y^2}\,dy\,dx. \tag{45}$$

If we multiply dy by 2, the first integration will come under (22), p. 493, the x of that formula being $2y$ here and the a^2 being $36-x^2$.

Hence the first integration gives

$$\tfrac{1}{6}\left[y\sqrt{36-x^2-4y^2}+\frac{36-x^2}{2}\sin^{-1}\frac{2y}{\sqrt{36-x^2}}\right]_0^{\frac{1}{2}\sqrt{36-x^2}}dx. \tag{46}$$

This reduces at once to $\frac{\pi}{24}(36-x^2)\,dx$, the volume of the slice shown in Fig. 82. The final integration then gives:

$$\tfrac{1}{8}V=\frac{\pi}{24}\left[36x-\tfrac{1}{3}x^3\right]_0^6=6\pi.$$

That is, the entire volume is 48π.

Remarks. (I) Any section of the ellipsoid perpendicular to the X-axis is an ellipse $4y^2+9z^2=36-x^2$, or

$$\frac{y^2}{\frac{1}{4}(36-x^2)}+\frac{z^2}{\frac{1}{9}(36-x^2)}=1. \tag{47}$$

The semi-axes are $\frac{1}{2}\sqrt{36-x^2}$ and $\frac{1}{3}\sqrt{36-x^2}$ and the area (πab) is $\frac{\pi}{6}(36-x^2)$. One-fourth of this, times dx, gives as the volume of a slice like that in Fig. 82 the same value as above: $\frac{\pi}{24}(36-x^2)dx$. Starting with this known, one integration suffices. But in many problems the cross section area is unknown and two integrations are necessary.

(II) Strictly speaking, the volume is the *limit* of the sum of many rectangular columns, $z\,\Delta y\,\Delta x$. But, as shown in the Appendix, p. 488, this limit of a sum is the double integral (42) obtained here by the idea of tiny elements.

§ 132. Cylindrical Elements. In regarding a solid as composed of slender columns, it is often best to use as the base of each column not a rectangular element $dx\,dy$ but a polar element $r\,dr\,d\theta$.

We then need to know the height z for any column in terms of r and θ. In other words, we virtually need the equation of the top surface in cylindrical coördinates. It is well, however, wherever feasible, to *see* from a drawing what the value of z is, rather than to depend upon some equation for it. Greater resourcefulness and independence can be gained in that way. The same remark holds when we are using rectangular coördinates, or any others.

Ex. I. A cylinder of diameter 8 in. cuts a sphere of radius 8 in. in such a way that one element passes through the center. Find the volume of that part of the sphere within the cylinder.

Fig. 83 a shows the upper half of the required volume, points on the intersection being located by means of planes through the element mentioned. Fig. 83 b shows the section made by one such cutting plane, when viewed directly or at right angles.

The volume of any column-element is $z\,r\,dr\,d\theta$. But by Fig. 83 *b*, $z=\sqrt{64-r^2}$, so that the element of volume is

$$\sqrt{64-r^2}\quad r\,dr\,d\theta.$$

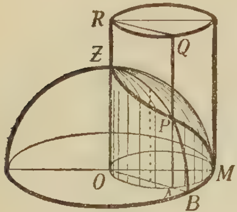


FIG. 83 *a*.

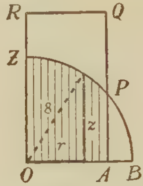


FIG. 83 *b*.

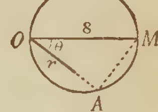


FIG. 83 *c*.

Looking directly down upon the base as in Fig. 83 *c*, the limits of integration are seen to be: for r, from 0 to $8\cos\theta$; for θ, from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, or from 0 to $\frac{\pi}{2}$ and double the result.

$$\therefore\quad \tfrac{1}{2}V=2\int_0^{\frac{\pi}{2}}\int_0^{8\cos\theta}\sqrt{64-r^2}\quad r\,dr\,d\theta,$$

$$=\frac{2}{3}\int_0^{\frac{\pi}{2}}\Big[-(64-r^2)^{\frac{3}{2}}\Big]_0^{8\cos\theta}d\theta=\frac{1024}{3}\int_0^{\frac{\pi}{2}}(1-\sin^3\theta)d\theta$$

$$=\frac{1024}{3}\Big[\theta+\cos\theta-\frac{1}{3}\cos^3\theta\Big]_0^{\frac{\pi}{2}}=\frac{1024}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right).$$

Doubling this gives the entire volume, above and below the base of Fig. 83 *a*.

Remarks. (I) It would be incorrect to run θ from 0 to π, instead of from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, — as may be seen from the base. (Fig. 83 *c*.)

(II) In finding this volume in rectangular coördinates, the column element $z\,dy\,dx$ would be $\sqrt{64-x^2-y^2}\,dy\,dx$. And the limits [as seen from the base circle $(x-4)^2+y^2=16$, or $x^2+y^2=8x$] would be: for y, from 0 to $\sqrt{8x-x^2}$, and double; for x, from 0 to 8. Thus

$$\tfrac{1}{2}V=2\int_0^8\int_0^{\sqrt{8x-x^2}}\sqrt{64-x^2-y^2}\,dy\,dx.$$

The integration is much more complicated than that above.

EXERCISES

1. Find the volume under the surface $z=x^2+y^2$ cut off by the coordinate planes and the planes $x=4$ and $y=2$.

2. Find the volume under the surface $z=xy+20$, above the plane $z=0$, and within the cylinder $x^2+y^2=25$:

(*a*) By rectangular elements; (*b*) By cylindrical elements.

3. Find the volume bounded by $2x+3y+5z=60$ and the coördinate planes.

4. (*a*) Find by single integration the volume bounded by $z=36-x^2-4y^2$ and the plane $z=0$. (*b*) Check by double integration.

5. (*a*), (*b*). The same as Ex. 4 (*a*), (*b*) for the upper half of the ellipsoid $x^2+4y^2+25z^2=100$.

6. Find by double integration in two ways the volume of an octant of a sphere of radius 6 ft. Check by elementary geometry.

7. Find the volume of a circular hole of radius 2 in. punched centrally through a sphere of radius 5 in.

8. A solid hemisphere of radius 10 in. is cut by a cylinder of radius 5 in., one of whose elements is perpendicular to the flat side of the hemisphere at the center. Find the common volume.

9. Find by double integration the volume of a cone of radius 6 in. and height 12 in. Check by elementary geometry.

10. A cylinder of height 20 and diameter 10 is cut by an inverted cone of height 20 and diameter 20, whose axis is one element of the cylinder. Find the volume of the cylinder outside the cone.

The following are somewhat more complicated in their details.

11. A cone of height 10 ft. and radius 5 ft. is cut by a vertical plane 3 ft. from its axis. Find the volume cut off. [Use cylindrical coördinates. To find the limits draw the base alone.]

12. Find the volume in the first octant bounded by the coördinate planes and the hypocycloidal surface $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}=a^{\frac{2}{3}}$. [Hint: In the first integration use the substitution $y=(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}\sin^3\phi$. Cf. § 106.]

13. Find the volume of a square hole, 4 in. by 4 in., punched centrally through a sphere of radius 4 in. [Use cylindrical coördinates and one-fourth the base, in the shape of an isosceles right triangle. Cf. (77), p. 497.]

14. Work Ex. 13 by using rectangular coördinates. [Cf. Ex. 3 (*c*), p. 186.]

§ **133. Surface Area.** The area of a curved surface, other than a surface of revolution, is usually found by double integration. Speaking freely: We regard the surface as composed of tiny portions, each so small as to be virtually flat and virtually to coincide with a small part of the tangent plane at the point. Consider any such portion dS of the surface, located directly above or below a rectangular element $dx\,dy$ of the base plane XOY. Then $dx\,dy$ is the projection of dS upon plane XOY. And if γ is the inclination of the tiny portion of surface considered, or of the tangent plane, we have by *Intro.*, § 113:

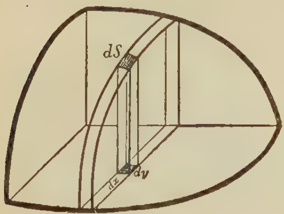


FIG. 84.

$$dx\,dy = dS \cos\gamma, \qquad \text{or} \qquad dS = dx\,dy \sec\gamma.$$

But by (70), p. 79, $\sec\gamma = \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}$.

$$\therefore \qquad dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy. \tag{48}$$

After finding these partial derivatives from the equation of the given surface and substituting in (48), we integrate. The limits must be such as to run the element $dx\,dy$ or $dy\,dx$ over the entire base, above which (or below which) the required surface area stands.

We have assumed here that the surface is not any kind of a cylinder with elements perpendicular to the plane XOY. Such a cylinder we could project upon another reference plane, say YOZ with

$$dS = \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz. \tag{49}$$

And $\partial x/\partial z$ would vanish, since z does not appear in the equation of a cylinder whose elements are parallel to the Z-axis.

It is not necessary that dS be so chosen that its projection will be a rectangular element $dx\,dy$. Sometimes a polar element $r\,dr\,d\theta$ is much preferable. In that case

$$dS = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\; r\,dr\,d\theta. \tag{50}$$

We find the partial derivatives as formerly, but before integrating transform the radical into a function of r and θ instead of x and y.

Ex. I. Find the area of the part of the sphere included within the cylinder in Fig. 83, p. 222.

The equation of the sphere is $x^2+y^2+z^2=64$, or

$$z=\sqrt{64-x^2-y^2}.$$

$$\therefore \frac{\partial z}{\partial x}=-\frac{x}{\sqrt{64-x^2-y^2}}, \qquad \frac{\partial z}{\partial y}=-\frac{y}{\sqrt{64-x^2-y^2}}.$$

$$\therefore \sec\gamma=\sqrt{1+\frac{x^2+y^2}{64-x^2-y^2}}=\frac{8}{\sqrt{64-x^2-y^2}}=\frac{8}{\sqrt{64-r^2}}.$$

To run the element $r\,dr\,d\theta$ over the circular base, the limits of integration are the same as in Ex. I, pp. 221–22.

$$\therefore\ S=2\int_0^{\frac{\pi}{2}}\int_0^{8\cos\theta}\frac{8\,r\,dr\,d\theta}{\sqrt{64-r^2}}.$$

Integrating we find

$$S=16\int_0^{\frac{\pi}{2}}\Big[-(64-r^2)^{\frac{1}{2}}\Big]_0^{8\cos\theta}d\theta=128\int_0^{\frac{\pi}{2}}(1-\sin\theta)d\theta$$
$$=64(\pi-2).$$

This gives only the upper area within the cylinder, and must be doubled.

Instead of regarding a small portion of a curved surface as flat, — according to the simple and free procedure above, — we can, in a strictly accurate manner, express the surface as the limit of a sum, and then as a double integral. [Cf. Appendix, p. 488.]

EXERCISES

1. Find the area of each of the following surfaces within a cylinder of radius 5, whose axis is the Z-axis:

(a) $z = xy$, (b) $2z = x^2 + y^2$.

2. Find by double integration the area of a hemisphere of radius 3 in. Check by elementary geometry.

3. The same as Ex. 2 for a cone of height 10 in. and radius 5 in.

4. Find the area of the sphere cut out in Ex. 7, p. 223.

5. Find the area of the part of the hemispherical surface included within the cylinder in Ex. 8, p. 223.

6. Find the area of the cone within the cylinder in Ex. 10, p. 223.

7. Find the area of the part of the cone cut off in Ex. 11, p. 223.

8. Two cylinders of radius 3 ft. intersect, with their axes perpendicular. Find the area of either within the other.

9. A square hole 4 in. by 4 in. is punched vertically through a horizontal cylinder of radius 5 in., two sides of the hole being parallel to, and 2 in. from, the axis of the cylinder. Find the area cut out.

10. In Ex. 9 find also the volume cut out.

11. Find the mass of a thin spherical shell of radius 3, if the surface density varies thus with the distance r from the vertical diameter: $D = 20 - r^2$.

12. In Ex. 11 find also the moment of inertia of the shell about the vertical diameter.

13. Find the mass of a solid sphere of radius 3 if the density at any point varies as in Ex. 11.

14. Find the area of $z = 10 - x^2 - 2y^2$ within the elliptic cylinder $x^2 + 4y^2 = 16$.

§ 134. Area within Any Bounding Surface. The foregoing areas have been inclosed by vertical planes or cylinders, which defined clearly the extent of the base above which each area stood. But, in general, when a required area is cut from one surface by another, we must ascertain for ourselves the proper base. If we *eliminate z by combining the equations of the two surfaces*, the resulting single equation will represent a vertical cylinder through the intersection of the two. (Cf. § 125.) Hence that resulting equation will also define the bounding curve in the base plane $z = 0$.

(We should, however, always examine the surfaces sufficiently to make sure that the area to be calculated does not bulge beyond the base of the projecting cylinder through the intersection.)

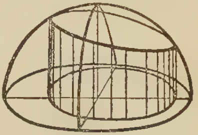


FIG. 85 *a*.

Aside from having to determine the base the procedure is as formerly. The partial derivatives are calculated, of course, from the equation of that surface whose area is desired, in whole or in part.

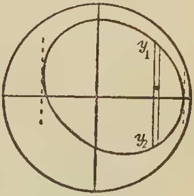


FIG. 85 *b*.

For purposes of comparison, the process under discussion will be illustrated by an example which can be worked more simply by another method. Even if you do not verify the more complicated details, follow the process carefully.

Ex. I. Find the area of that part of the plane

$$z = 8 - .3\,x + .2\,y \tag{51}$$

which lies within the sphere $x^2+y^2+z^2 = 100$.

From (51) we find $\partial z/\partial x = -.3$, $\partial z/\partial y = .2$. Hence

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1.13}. \tag{52}$$

Eliminating z from the equations of the plane and sphere gives for the projecting cylinder (Fig. 85 *a*):

$$1.09\,x^2 - .12\,xy + 1.04\,y^2 - 4.8\,x + 3.2\,y - 36 = 0. \tag{53}$$

The base of this cylinder may be plotted and the limits of integration found, by solving (53) for y as a function of x.

First rearranging according to powers of y we have the quadratic:

$$1.04\,y^2 \quad + (3.2 - .12\,x)y \quad + (1.09\,x^2 - 4.8\,x - 36) = 0.$$

The formula, $y = (-B \pm \sqrt{B^2 - 4\,AC}) \div 2\,A$, gives on simplifying:

$$y = \frac{.06\,x - 1.6 \quad \pm\sqrt{40 + 4.8\,x - 1.13\,x^2}}{1.04}. \tag{54}$$

As long as the radical is real, and distinct from zero, there are two values of y for each value of x. (The + sign gives y_2 and the − sign y_1 in Fig. 85 *b* as seen from above.) These are the y-limits in the first integration.

At the extreme values of x, y_1 and y_2 become equal; and hence

$$40+4.8\,x-1.13\,x^2=0. \tag{55}$$

The roots of this, x_1 and x_2, are the limits of the second integration.

$$x=\frac{4.8\pm\sqrt{203.84}}{2.26}=8.44,\ -4.19,\ \text{approx.}$$

$$\therefore\ S=\int_{-4.19}^{8.44}\int_{y_1}^{y_2}\sqrt{1.13}\,dy\,dx$$

$$=\sqrt{1.13}\int_{-4.19}^{8.44}\frac{2\sqrt{40+4.8\,x-1.13\,x^2}}{1.04}\,dx.$$

Using formula (40), p. 495, this gives

$$S=\frac{\sqrt{1.13}}{1.04}\left[\left(x-\frac{2.4}{1.13}\right)\sqrt{40+4.8\,x-1.13\,x^2}\right.$$
$$\left.+\frac{50.96}{(1.13)^{\frac{3}{2}}}\sin^{-1}\frac{1.13\,x-2.4}{\sqrt{50.96}}\right]_{-4.19}^{8.44}$$

The first radical vanishes at both limits, and the arcsine is $\pi/2$ at one limit and $-\pi/2$ at the other. Hence:

$$S=\frac{50.96\,\pi}{(1.04)(1.13)}=\frac{49\,\pi}{1.13}. \tag{56}$$

Remarks. (I) Since the curve of intersection of the given sphere and plane must be a circle, the required area could be found if we knew the radius r of the circle. The latter can be found by using trigonometry, but more easily by a later method. (Chap. XII.)

(II) It would be well to reread this entire article, noting especially the general plan, rather than the details in Ex. I.

§ **135. Further Volumes.** Two surfaces, one below the other, may curve up or down so as to meet all around and inclose a space. This space can be regarded as composed of thin columns reaching from the lower surface to the upper. The base under any column is $dx\,dy$ or $r\,dr\,d\theta$. The height is (z_2-z_1), *i.e.*, the difference of the heights of the two sur-

faces at the column considered. Hence the volume of the inclosed space is

$$V = \int\int (z_2 - z_1) dx\, dy. \tag{57}$$

The limits of integration are obtained, as for area, by considering the portion of the plane XOY, above which the required volume is located. A projecting cylinder through the curve of intersection will give the limits if the volume does not bulge beyond the intersection.

EXERCISES

1. (*a*) Find the area of the paraboloid $2z = x^2 + y^2$ below the plane $z = 10$.

(*b*) Find the volume between the paraboloid and the plane.

2. (*a*) Find the area of the part of the surface $z = x^2 + y^2$ which lies below the surface $z = 18 - x^2 - y^2$.

(*b*) Find the volume bounded by these two surfaces.

3. Find the area of the part of the plane $z = 2x$ which lies within (*a*) The ellipsoid $x^2 + 9y^2 + 2z^2 = 36$; (*b*) The paraboloid $z = x^2 + y^2$.

4. Like Ex. 3 for $z = 8x + 4y + 8$ and $z = 4x^2 + y^2$.

5. (*a*)–(*c*). Find the volume bounded by each pair of surfaces in Ex. 3 (*a*), 3(*b*), 4, respectively.

6. Find the area of the part of the plane $z = x + 2y + 5$ within the sphere $x^2 + y^2 + z^2 = 25$.

7. Like Ex. 6 for $z = 5 + y - \frac{1}{2}x$ and $x^2 + y^2 + z^2 = 100$.

8. (*a*), (*b*). Find the volume bounded by each pair of surfaces: $z = y^2$, $z = 8 - 2x^2 - y^2$; $z = \sqrt{2(x^2 + y^2)}$, $z = 4 - x^2 - y^2$.

9. Find the area of the sphere cut out in Ex. 13, p. 223.

10. Two cones, of height 30 ft. and radius 30 ft., intersect so their axes are 30 ft. apart. Find their common volume. [Cf. Ex. 11, p. 223.]

11. Use the result of Ex. 10 to answer this question: A pile of sawdust has the shape of three overlapping cones, each of height 30 ft. and radius 30 ft., with axes 30 ft. apart and in one plane. What is the total volume of the pile?

§ 136. Centroid of a Thin Shell. As in the case of a flat plate, the centroid of a thin shell is found by equating two

torques. The weight of each minute element of the shell has a torque about any chosen horizontal axis; and the total of these tiny torques must equal the actual torque which the entire weight of the shell would have if concentrated at the centroid.

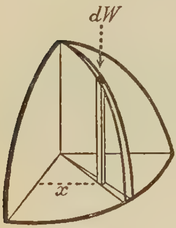


FIG. 86.

If the chosen axis be the Y-axis, and dW be the weight of any tiny element dS of the shell, acting parallel to the Z-axis, its arm will be x, and its torque $x\,dW$. The total of these small torques is then

$$T = \int\!\!\int x\,dW.$$

But the torque due to the entire weight W applied at the centroid with arm $\bar{x}$ is $T = W\bar{x}$. Equating values of T, and remembering that mass and weight are proportional we may write

$$\bar{x} = \frac{\int\!\!\int x\,dW}{W} = \frac{\int\!\!\int x\,dM}{M}. \tag{58}$$

Similarly, taking torques about the X-axis we find

$$\bar{y} = \frac{\int\!\!\int y\,dM}{M}. \tag{59}$$

Further, if we turn the shell and axes so that the vertical pull of gravity would be parallel to the X-axis or Y-axis, we find that

$$\bar{z} = \frac{\int\!\!\int z\,dM}{M}. \tag{60}$$

To apply (58)–(60) we must first express dM in terms of coördinates. If D be the surface density, and $\sec\gamma\,dy\,dx$ (or $\sec\gamma\,r\,dr\,d\theta$) be the element of surface area, we may write

$$dM = D\sec\gamma\,dy\,dx, \qquad \text{or} \qquad Dr\sec\gamma\,dr\,d\theta.$$

Then, after multiplying dM by x, y, or z (or by the equivalent in cylin-

drical coördinates), we integrate. The limits are found in the base as formerly.

At all elements of the shell or surface, z has a definite value in terms of x and y, or r and θ; and that value is to be substituted for z in (60) before integrating.

EXERCISES

1. A thin curved plate has the shape of the parabolic cylinder $z=9-x^2$ in the principal octant, from $y=0$ to $y=10$. Its surface density varies thus: $D=20+y$. Find its mass.

2. Find $\bar{x}$, $\bar{y}$, and $\bar{z}$ for the centroid of the plate in Ex. 1.

3. Find the centroid of a hemispherical shell of radius 20 in., if the surface density varies as the distance r from the axis of symmetry.

4. Find the centroid for a conical shell of radius 10 in. and vertical height 20 in. if the surface density varies thus with the distance r from the axis: $D=10+r$.

5. Find the moment of inertia of the shell in Ex. 4 about its axis.

6. A funnel has the shape of a cone with vertex angle 90°, cut off by planes 2 cm. and 20 cm. from the vertex. The surface density is constant. Find the centroid.

7. Find the centroid of the part of the paraboloid $z=xy$ inclosed by a cylinder of radius 4 whose axis is the Z-axis.

8. The density of a solid hemisphere of radius 10 varies thus with the distance r from its axis of symmetry: $D=20-r$. If a circular hole of radius 3 is punched through centrally, perpendicular to the flat side, what will be the moment of inertia of the remaining portion with respect to the axis of symmetry?

9. Find the moment of inertia about the Z-axis of a homogeneous solid bounded by the surfaces $z=x^2+y^2$ and $z=50-x^2-y^2$.*

PART III. TRIPLE INTEGRATION

§ 137. Ultimate Elements of Volume. In many physical problems we have to deal with quantities which vary from point to point in a solid and cannot be considered as constant even along a slender column-element. We then have to divide up the column into still more minute elements, or

* A homogeneous solid is one of constant density.

particles, — of height dz, and base $dy\,dx$ or $r\,dr\,d\theta$. The volume of such a particle or ultimate element of volume is then

$$dz\,dy\,dx, \qquad \text{or} \qquad r\,dz\,dr\,d\theta. \tag{61}$$

Three successive integrations are needed in "summing up."

Ex. I. Express the mass, in the principal octant, of a solid ellipsoid $3x^2+4y^2+9z^2=36$, if the density D varies with x, y and z in some way.

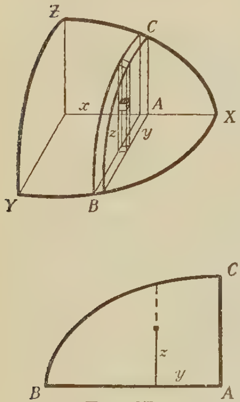


FIG. 87.

Regarding D as constant for a particle, the mass of the latter is

$$dM = D\,dz\,dy\,dx. \tag{62}$$

Integrating first with respect to z would give the mass of a slender vertical column; next, with respect to y, a thin slice; and finally, with respect to x, the whole solid. (Fig. 87.)

The limits of integration are: for z, zero and the value of z at the surface in general, viz.

$$\tfrac{1}{3}\sqrt{36-3x^2-4y^2};$$

for y, zero and the value of y at the boundary of the base, viz.

$$\tfrac{1}{2}\sqrt{36-3x^2};$$

for x, zero and the extreme value on the X-axis, viz. $\sqrt{12}$.

$$\therefore \qquad M = \int_0^{\sqrt{12}}\int_0^{\frac{1}{2}\sqrt{36-3x^2}}\int_0^{\frac{1}{3}\sqrt{36-3x^2-4y^2}} D\,dz\,dy\,dx. \tag{63}$$

The integrations cannot be carried out until we know just what function of x, y, and z the density D is.

More generally, to get the limits for a volume between a lower and an upper surface, first run z from its value at the lower surface (usually in terms of x and y or r and θ) to its value at the upper. Then eliminate z from the two equations, or otherwise get the equation of the projecting cylinder, and from the latter equation find the y and x limits as in §§ 134–35.

§ 138. Centroid of a Solid. If dM is the mass of any particle in a solid, and x, y, z are its coördinates, the centroid of the solid is given by:

$$\bar{x}=\frac{\int\int\int x\,dM}{M},\quad \bar{y}=\frac{\int\int\int y\,dM}{M},\quad \bar{z}=\frac{\int\int\int z\,dM}{M}. \tag{64}$$

To carry out the integrations we take

$$dM=D\,dz\,dy\,dx, \qquad \text{or} \qquad dM=D\,r\,dz\,dr\,d\theta,$$

according as rectangular or cylindrical coördinates appear the simpler. The limits are best seen by drawing sections. (Cf. Fig. 87 above, Fig. 83, p. 222, and Figs. 88, 89 below.)

The proof of (64) is exactly that given in § 136 for a thin shell or surface, except that triple integration is needed to sum up for all particles in a solid. And, of course, in Fig. 86, p. 230, the elementary particle can now be anywhere along the vertical column shown, rather than merely at the top.

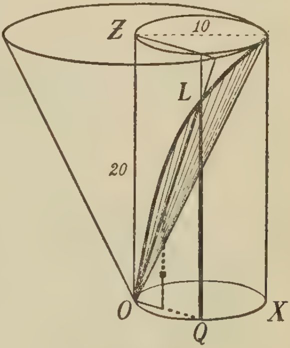


Fig. 88.

Because of the latter fact, *the value of z at the upper surface must not be substituted for z in* (64) *before integrating.* The z in (64) belongs to each and every particle, all the way up the column.

§ 139. Moment of Inertia. If dM is the mass of any particle in a solid, and R its distance from any chosen line, its moment of inertia about that line as axis is $R^2\,dM$. The total moment of inertia for the entire solid is

$$I=\int\int\int R^2\,dM. \tag{65}$$

Here again, dM is either $D\,dz\,dy\,dx$ or $D\,r\,dz\,dr\,d\theta$. And, as seen in § 124, if the chosen axis is the Z-axis, then

$$R^2=x^2+y^2, \quad \text{or} \quad R^2=r^2; \tag{66}$$

while, if the chosen axis is the X-axis, then

$$R^2=y^2+z^2, \quad \text{or} \quad R^2=r^2\sin^2\theta+z^2. \tag{67}$$

(What is R^2 if the chosen axis is the Y-axis?)

Ex. I. A solid cylinder of height 20 in., diameter 10 in., and varying density D is cut by an inverted cone of like height and of radius 10 in., whose axis is an element of the cylinder. Choose axes conveniently, and express as definite integrals the moment of inertia about the Z-axis, and also $\bar{x}$ and $\bar{z}$, for the part of the cylinder outside the cone. (Fig. 88.)

Using cylindrical coördinates, the mass is

$$M=\iiint D\, r\, dz\, dr\, d\theta.$$

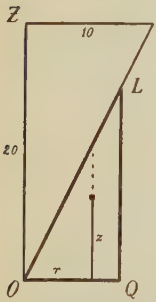


FIG. 89 *a*.

To determine the limits observe that θ is to be constant during the first two integrations; and draw a section $ZOQL$ accordingly. The first two integrations will sum up elements in this section alone. Then draw a column, up which z may vary while r is constant. Fig. 89 *a* shows this section, viewed directly.

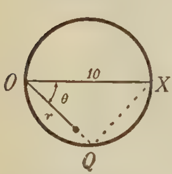


FIG. 89 *b*.

By similar triangles the upper z limit is $z=2r$. A direct view of the base, as in Fig. 89 *b*, shows the r limits to be 0 and $10\cos\theta$, and the θ limits to be $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$$\therefore \qquad M=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{10\cos\theta}\int_0^{2r} Dr\, dz\, dr\, d\theta. \tag{68}$$

Likewise I_z, $\bar{z}$, and $\bar{x}$ are given, respectively, by the following integrals:

$$\iiint Dr^3\, dz\, dr\, d\theta,$$

$$\frac{\iiint Dr\, z\, dz\, dr\, d\theta}{M}, \qquad \frac{\iiint Dr^2\cos\theta\, dz\, dr\, d\theta}{M},$$

where the limits of integration are the same as for M in (68).

EXERCISES

1. Find the volume bounded by the paraboloid $z = x^2 + 4y^2$, the three coördinate planes, and the planes $x = 3$ and $y = 2$. Express this both as a triple and as a double integral.

2. Find the centroid of the volume in Ex. 1. Can this be expressed by means of double integrals?

3. Like Ex. 1 for $z = xy$ and the planes $z = 0$, $x = 3$, and $y = 2$.

4. For the volume in Ex. 3 determine the centroid:

(*a*) Find $\bar{x}$; (*b*) Find $\bar{y}$; (*c*) Find $\bar{z}$.

5. Find the volume bounded by $z = x^2 + y^2$ and $z = 16$. Express this as a triple, a double, a single integral.

6. Find $\bar{z}$ for the volume in Ex. 5. (Use any method.)

7. A solid has the shape of the volume in Ex. 5, with a varying density, $D = 10 + .6\,z$. Find its moment of inertia about the Z-axis.

8. (*a*) Find the centroid of the right-hand half of the solid ellipsoid

$$\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1.$$

(*b*) Find also the moment of inertia with respect to the Y-axis.

9. In a cone of height 12 and radius 4 the density varies with the distance z from the base: $D = 6 - .3\,z$.

(*a*) Find the mass M, by triple integration.
(*b*) Find $\bar{z}$ for the centroid, in like manner.
(*c*) Find I_z, the moment of inertia about the axis of the cone.
(*d*) Find M also by single integration. Could $\bar{z}$ be so found? I_z?

10. In Ex. I, § 139, take $D = k$, and calculate

(*a*) M; (*b*) I_z; (*c*) $\bar{z}$; (*d*) $\bar{x}$.
(*e*) Also find I_x. (*f*) By inspection, what is $\bar{y}$?

11. In Ex. I above, find the volume of the part of the cylinder inside the cone. Check with Ex. 10 (*a*) by geometry.

12. In Ex. I above, take $D = 10 - .2\,z$, and calculate I_z.

13. A homogeneous sphere of radius 10 in. is cut by a cylinder of radius 5 in., one of whose elements is a diameter of the sphere ZZ'. For the portion of the sphere inside the cylinder, find:

(*a*) The mass; (*b*) Location of the centroid; (*c*) I_z.

§ 140. Spherical Coördinates. There is still another system of coördinates, which is especially convenient in many problems relating to a sphere or cone. This system is based

upon a vertical axis OZ, drawn upward from an origin O, and upon a vertical reference plane XOZ through OZ. (Fig. 90.)

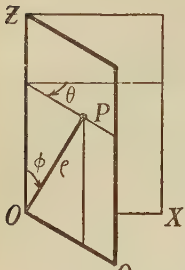


FIG. 90.

Any point P in space lies in some vertical plane QOZ. The location of P is known if we know $\angle\theta$ which QOZ makes with XOZ, $\angle\phi$ which line OP makes with OZ, and the distance ρ of P from O. These quantities (ρ, ϕ, θ) are called the spherical coördinates of P.

Where are all points in space for which θ is constant? All for which ϕ is constant?

Evidently all points for which ρ is constant lie on a sphere. If θ is also constant, they lie on a meridian of the sphere; or if ϕ is constant, on a latitude circle of the sphere.

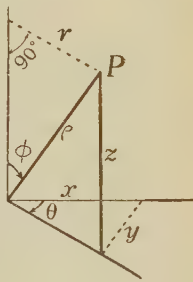


FIG. 91.

Taking the point P anywhere in space, ρ may have any positive value; ϕ any value from $0^{(r)}$ to $\pi^{(r)}$ (the angles between $\frac{\pi^{(r)}}{2}$ and $\pi^{(r)}$ belonging to points below the plane XOQ); and θ any value.

Relation to other systems. From Fig. 91 we see that

$$\boldsymbol{r=\rho\sin\phi,\qquad z=\rho\cos\phi.} \tag{69}$$

Also, since $x=r\cos\theta$ and $y=r\sin\theta$,

$$\therefore\qquad x=\rho\sin\phi\cos\theta,\qquad y=\rho\sin\phi\sin\theta. \tag{70}$$

By these relations we can change any cylindrical or rectangular expression into spherical coördinates. We note also that

$$\rho^2=r^2+z^2 = x^2+y^2+z^2. \tag{71}$$

The distance of any point P from the Z-axis is r or $\rho\sin\phi$; while the distance from the X-axis is $\sqrt{y^2+z^2}$ or $\sqrt{\rho^2-x^2}$, which by (70) becomes

$$\rho\sqrt{1-\sin^2\phi\cos^2\theta}.$$

The relations (69) should be memorized. The others we should be able to derive quickly when needed.

§ 141. Elements for Spherical Coördinates. Any solid may be decomposed into tiny elements as follows: First pass two vertical planes through the axis OZ in nearly the same direction, — say at angles θ and $\theta+d\theta$, — cutting out a thin wedge-shaped slice, as in Fig. 92 *a*. Next pass two conical surfaces, at angles ϕ and $\phi+d\phi$ from OZ, cutting from the slice a pointed or tapering column. Finally let two spherical surfaces with radii ρ and $\rho+d\rho$ cut from the column a tiny block or particle. (Fig. 92 *b* shows the column and block as seen from the left.)

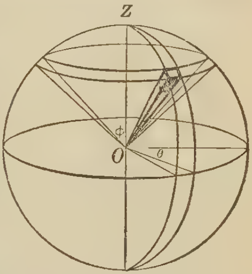


FIG. 92 *a*.

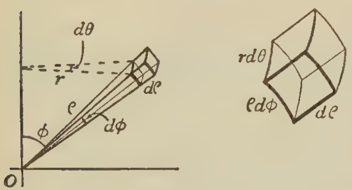


FIG. 92 *b*.

The ultimate element thus obtained we regard as a tiny rectangular solid. One dimension is $d\rho$; another, a circular arc with radius ρ and central angle $d\phi$, is $\rho d\phi$; and the third, an arc with radius r and central angle $d\theta$, is $r\,d\theta$ or $\rho\sin\phi\,d\theta$. Thus the element of volume is $dV=d\rho\cdot\rho d\phi\cdot\rho\sin\phi\,d\theta$, or

$$dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta. \tag{72}$$

To find a centroid or moment of inertia we may use the same formulas as formerly; *e.g.*,

$$\bar{x}=\frac{\iiint x\,dM}{M}, \qquad I_z=\iiint r^2\,dM.$$

But we must now express x, r, etc., in terms of ρ, ϕ, θ, by means of (70) and (69); and replace dM by the density D times dV as given by (72).

Ex. I. A solid sphere of constant density k, and radius 20 in., is cut by a cone whose axis OZ is a diameter of the sphere and whose vertex O lies on the surface. The half angle of the cone is 30°. Find the mass, centroid, and the moment of inertia with respect to the diameter in question, of that part of the sphere which lies within the cone. (Fig. 93 *a*.)

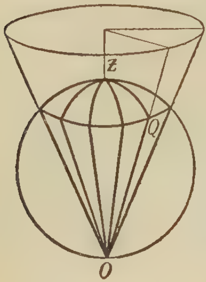


FIG. 93 *a*.

(*A*) Let us use spherical coördinates, with the origin at O, and OZ as Z-axis. Then

$$dM = k\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta. \tag{73}$$

The first two integrations will keep θ constant and hence will operate in a section OZQ, shown directly in Fig. 93 *b*. The first integration, with respect to ρ, sums the masses of particles having a constant ϕ; *i.e.*, particles along a line or pointed column. The upper limit, clearly different for different lines, is $\rho = 40 \cos\phi$. The second integration, with respect to ϕ, sums the masses for all such columns in the section or slice OZQ. The limits are 0 and $\frac{\pi}{6}$. In the last integration, section OZQ swings around, from $\theta = 0$ to $\theta = 2\pi$.

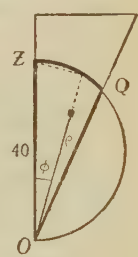


FIG. 93 *b*.

$$\therefore M = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{40\cos\phi} k\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{k(40^3)}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{6}} \cos^3\phi \sin\phi \, d\phi \, d\theta$$

$$= \frac{16000\,k}{3}\int_0^{2\pi}\Big[-\cos^4\phi\Big]_0^{\frac{\pi}{6}} d\theta.$$

Now $\cos^4\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^4 = \frac{9}{16}$ and $\cos^4 0 = 1$. Hence finally

$$M = \frac{7000\,k}{3}\int_0^{2\pi} d\theta = \frac{14000\,\pi k}{3}. \tag{74}$$

(*B*) Evidently the centroid lies on OZ. Thus $\bar{x}=\bar{y}=0$ while

$$\bar{z}=\frac{\iiint z\,dM}{M}=\frac{\iiint(\rho\cos\phi)k\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}{M}.$$

The limits are as above. Hence, using the value of M already found:

$$\bar{z}=\frac{3}{14000\,\pi k}\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{40\cos\phi}k\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta.$$

Carried out, these integrations give finally

$$\bar{z}=\frac{3}{14000\,\pi k}\cdot\frac{370000\,\pi k}{3}=26\tfrac{3}{7}. \tag{75}$$

Considering the shape of the solid in question, this is about what we should expect for the height of the centroid.

(*C*) For the moment of inertia about OZ we have

$$I_z=\int\int\int r^2\,dM=\int\int\int k\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta, \tag{76}$$

with the same limits as above. Worked out, this gives

$$I_z=\frac{1\,340\,000\,\pi k}{3}. \tag{77}$$

Remark. This gives for the radius of gyration (Cf. Ex. 14 below), $R=\sqrt{670/7}=9.78$, approx. Since the distance of Q from OZ is found by trigonometry to be $10\sqrt{3}$ or 17.32, this result appears reasonable.

In many cases it is possible to integrate first with respect to θ from 0 to 2π by merely introducing a factor 2π, and thus dispose of one integral immediately. Could this have been done in the example above?

Not only the shape of a solid but also the complexity of the expressions to be integrated should determine the system of coördinates to be used in a calculation.

EXERCISES

1. Find the volume of a sphere of radius a by using spherical coördinates with the origin at the center:

(*a*) Integrating first with respect to ρ;

(*b*) Integrating first with respect to ϕ.

2. Like Ex. 1 with the origin on the surface.

3. In a hemisphere of radius a, the density varies as the distance from the center: $D=k\rho$. Find

(a) The moment of inertia about the radius of symmetry;

(b) The radius of gyration with respect to the same axis;

(c) The position of the centroid.

4. (a)–(c). Like Ex. 3 (a)–(c), if $D=k$, a constant.

5. In an inverted cone, of height 20 cm. and vertex half-angle 30° the density varies as the distance from the vertex. Find

(a) The moment of inertia about the axis of the cone;

(b) The radius of gyration with respect to the same axis;

(c) The position of the centroid.

6. (a)–(c). Like Ex. 5 (a)–(c), if $D=k$, a constant.

7. Find the moment of inertia of a homogeneous cone of height 12 and radius 4 with respect to its own axis:

(a) Using spherical coördinates; (b) Using cylindrical coördinates.

8. A homogeneous sphere of radius a is cut by a cone of vertex angle 90°, whose vertex is at the center. Locate the centroid of the part of the sphere within the cone.

9. A homogeneous sphere of diameter 20 is cut by an unending cone of vertex angle 60°, whose vertex lies on the surface, and whose axis passes through the center. For the part of the sphere within the cone:

(a) Find the moment of inertia with respect to that axis;

(b) Locate the centroid.

10. (a), (b). Like Ex. 9 (a), (b), if D varies thus with the distance ρ from the vertex: $D=k/\rho$.

11. In Ex. 9 find the volume of the sphere outside the cone.

12. A sphere S of radius 20 is cut by a sphere of radius 10, whose center lies on the surface of S. Find the volume of the larger sphere within the smaller.

13. A cone whose vertex angle is 40° is just inscribed in a sphere of diameter 10 in. Express as definite integrals, without working out:

(a) The moment of inertia of the cone about its own axis;

(b) The coördinate $\bar{z}$ which locates the centroid.

14. From (74) and (77) find the radius of gyration mentioned in the *Remark* above.

§ 142. Rotational Energy. As previously seen (§ 84), the kinetic energy of a particle, dM, moving with any speed

V is $\frac{1}{2} V^2\, dM$. When a solid rotates about some axis OZ, with some angular speed ω rad./sec., any particle at a distance R from OZ has a linear speed $V = R\omega$, and hence energy $\frac{1}{2} R^2\omega^2\, dM$. The total rotational energy of the solid can be found by triple integration, after expressing R and dM in terms of whatever coördinates are most convenient.

If a solid has both translatory and rotary motion, then, as is shown in treatises on Mechanics, its total kinetic energy at any instant is the sum of two parts: (1) Energy of translation, which equals $\frac{1}{2} M$ times the square of the speed of the centroid at that instant; and (2) Energy of rotation, found by regarding every particle as momentarily revolving about an axis through the centroid.

The details of this are too involved to discuss in the present text.

§ 143. Attraction of a Solid. To find the gravitational attraction of a solid upon an exterior particle Q, of mass m, we first consider a particle P, of mass dM, anywhere in the solid. (Fig. 94.) As in § 120, the pull of P upon Q is

$$dF = \frac{G m\, dM}{\overline{QP}^2}. \qquad (78)$$

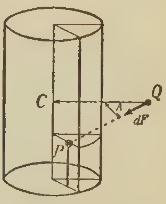


FIG. 94.

Different particles pull Q in different directions; and to some extent offset one another. We must, therefore, as formerly, find the component of dF in some direction; and then sum up such components for all particles dM in the solid.

If we can see by inspection in what direction the resultant pull must act, we need merely take components along that one direction. Otherwise we must get components along three independent directions; and, after integrating, combine to find the resultant.

Choosing the origin at Q and one axis in the direction considered will often simplify matters.

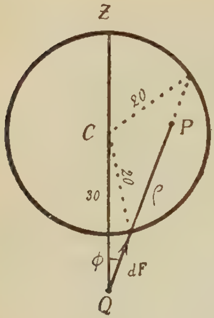


FIG. 95.

Ex. I. Find the attraction of a sphere of constant density k, and radius 20, upon a particle Q located 30 units from the center.

Fig. 95 gives a direct view of any section through the line joining Q to the center of the sphere. Choosing axes as shown, the resultant pull must be in the Z-direction (toward the center). The tiny part of this, dF_z, produced by dF equals $dF \cdot \cos \phi$. That is,

$$dF_z = \frac{G m dM}{\rho^2} \cos \phi = Gmk \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta. \qquad (79)$$

For either ρ limit, the Cosine Law gives

$$20^2 = \rho^2 + 30^2 - 60 \rho \cos \phi. \qquad (80)$$

$$\therefore \qquad \rho = 30 \cos \phi \pm \sqrt{400 - 900 \sin^2 \phi}. \qquad (81)$$

The + sign gives the upper limit ρ_2, and the − sign the lower limit ρ_1. Observe that $\rho_2 - \rho_1$ is simply twice the radical.

At the extreme value of ϕ, $\rho_1 = \rho_2$. *I.e.*, $400 - 900 \sin^2 \phi = 0$.

$$\therefore \qquad \phi = \sin^{-1} \tfrac{2}{3}. \qquad (82)$$

[Verify this by trigonometry, when ρ becomes tangent.]

Hence the total or resultant attraction is

$$F_z = Gmk \int_0^{\sin^{-1}\frac{2}{3}} \int_{\rho_1}^{\rho_2} \int_0^{2\pi} \sin \phi \cos \phi \, d\theta \, d\rho \, d\phi$$

$$= 2 \pi Gmk \int_0^{\sin^{-1}\frac{2}{3}} 2\sqrt{400 - 900 \sin^2 \phi} \sin \phi \cos \phi \, d\phi. \qquad (83)$$

The next integration comes under $u^{\frac{1}{2}} \, du$, with $u = 400 - 900 \sin^2 \phi$.

$$\therefore \quad F_z = \frac{\pi Gmk}{675} \Big[-(400 - 900 \sin^2 \phi)^{\frac{3}{2}} \Big]_0^{\sin^{-1}\frac{2}{3}} = \frac{320}{27} \pi Gmk.$$

Remark. This result is the same as if the entire mass of the sphere, $32000\pi k/3$, were concentrated at its center, 30 units away from Q. It was by a somewhat similar integration that Newton proved that any homogeneous sphere attracts as if concentrated at its center or centroid. The same is not true, however, for most solids.

EXERCISES

In Ex. 1–6, find the kinetic energy of each solid, rotating with angular speed ω about the axis named.

1. A homogeneous sphere of radius a, about a diameter.

2. Like Ex. 1 if the density varies inversely as the distance from the center: $D = k/\rho$.

3. A cylinder of height h and radius a, about its own axis, if D varies as the distance z from one base.

4. Like Ex. 3, about an element as axis.

5. A cone of height 6 in. and radius 2 in., about its own axis, if D varies as the distance ρ from the vertex.

6. A timber 100 in. long, with a constant triangular cross section whose sides are 3 in., 4 in., and 5 in., about the sharpest lengthwise edge.

In Ex. 7–13, find the attraction of each solid upon a particle Q, of mass m, located as specified.

7. Homogeneous hemisphere, radius a; Q at the center.

8. Like Ex. 7 if D varies as ρ, the distance from the center.

9. Homogeneous sphere, radius 10; Q on the surface.

10. Like Ex. 9, with Q 15 units from the center:

(*a*) Use spherical coördinates, with the origin at Q;

(*b*) The same, with the origin at the center;

(*c*) Use cylindrical coördinates with the origin at the center.

11. Homogeneous cone, of height 8 and radius 4; Q at the vertex.

12. Like Ex. 11, with Q on the axis, 2 units beyond the vertex.

13. Homogeneous cylinder, of height 8 and radius 4; Q on the axis, produced 4 units.

§ **144. Attraction of a Shell.** It is sometimes necessary to find the attraction exerted on a particle by a very thin shell; or upon a concentrated electrical charge by a charge dis-

tributed over a curved surface. The process is the same as for the attraction of a solid, with one exception: The attracting element of mass or charge is confined to a surface, and equals the surface element of area

$$\sec\gamma\,dy\,dx, \qquad \text{or} \qquad \sec\gamma\,r\,dr\,d\theta, \qquad \text{etc.,}$$

multiplied by surface density or charge density. Double integration therefore suffices instead of triple. (Cf. § 120.)

An element of spherical surface is expressible in spherical coördinates very simply if the origin is taken at the center. (Cf. Ex. 3 below.) The distance between attracting particles or point charges will then not be ρ, but will be given by formula (25), p. 205, or by the Cosine Law.

§ 145. Remarks on Chapter V. In concluding this chapter a few general observations should be made.

(I) *Multiple Integrals.* To find the attraction of one solid upon another six integrations may be required: three to get the attraction of the first solid upon any particle of the other, and three more to sum up for all particles of the second solid. In fact, it may be necessary to perform six integrations for each of three independent components of the resultant.

In the kinetic theory of gases eight successive integrals are required to express certain important quantities; and in many other scientific problems multiple integrals are encountered.

(II) *Choice of Elements.* Before setting up a triple integral it is well to inquire whether the desired quantity can be found readily by double or even single integration. Where the larger element is known, we not merely save a step by using it but sometimes avoid a hard preliminary integration involving tedious reductions.

(III) *Exact Arguments.* The free and convenient method of setting up integrals by considering variables as constant

in small portions of a surface or solid can be replaced by exact though longer arguments, if one reasons about the limit of a sum in each case.

In the Appendix, p. 487, theorems are proved connecting limits of sums, of certain types, with integrals. These provide a means of expressing quantities as definite integrals by an exact argument, — in most of the cases thus far considered. The specific application to several typical cases is pointed out. Ideas not covered there, but taken for granted in the course, are justified in advanced courses in analysis.

It may be profitable at this point to study carefully the discussion in the Appendix. Heretofore it has seemed best to concentrate our attention and energy upon gaining an insight into the free and convenient mode of analysis, and developing some facility and confidence in its use, — with only an occasional reminder that the idea of *the limit of a sum* contains the basic truth, for which our easy method is but an abbreviation.

(IV) *The Next Step.* Before making further applications of integration we need to consider certain methods of approximation, both in calculating functions by series and otherwise, and in integrating.

EXERCISES *

In Ex. 1–4, find the attraction of each shell of constant surface density upon a particle Q, of mass m, as specified.

1. Cone $z = 4 - \sqrt{x^2 + y^2}$ above $z = 0$; Q at the origin.

2. Cone of height 10 and radius 5; Q at the vertex.

3. Hemisphere of radius a; Q at the center. [Hint: Express the element of surface in spherical coördinates by omitting $d\rho$ from the volume element and putting $\rho = a$.]

4. Spherical shell of radius 10; Q 15 units from the center.

5. Express as a definite integral the kinetic energy of the first octant of the solid ellipsoid $4x^2 + 9y^2 + 16z^2 = 3600$, rotating about the Y-axis, with angular speed ω.

* For review exercises on Chapters I–V, see pp. 479 ff.

6. In a rectangular solid 20 ft. high, with a square base 2 ft. by 2 ft., the density varies as the distance above the base. Find the kinetic energy if this rotates with angular speed ω about one of the vertical edges.

7. A gas is expanding spherically. The pressure on the container varies with the volume. Express the total force on the container and the work done while the radius increases by $d\rho$. Show that the total work during any expansion can be expressed as $\int p\,dv$.

8. A sphere of diameter 20 is cut by a cone of vertex angle 60°, whose vertex O lies on the surface and whose axis passes through the center. It is also cut by a sphere of radius 5 with center at O. Find the volume of the larger sphere, outside the smaller and inside the cone.

9. In Ex. 8 find the volume inside both spheres and outside the cone.

10. A sphere of any radius is cut by a cylinder of half as great radius, one of whose elements is a diameter of the sphere. Find

(*a*) The area of the sphere within the cylinder;
(*b*) The volume of the sphere within the cylinder;
(*c*) The area of the cylinder within the sphere.

11. Find the volume bounded by the surfaces $x^2+y^2=4z$ and $x^2+y^2=z^2$. If feasible, express this volume as a single, a double, and a triple integral. Work out the value by using any one of these.

12. The surface density of a flat circular plate of radius a varies thus with the distance r from a point O on the circumference: $D=10+.6\,r$. Express as definite integrals the following quantities:

(*a*) A coördinate locating the centroid;
(*b*) Moment of inertia about the diameter of symmetry;
(*c*) Polar moment of inertia for an axis through O.

13. Plot the part of the surface

$$(x^2+y^2+z^2)^2=10\,xyz$$

which lies in the first octant. [Hint: First transform to spherical coördinates.]

14. Find the volume within the surface plotted in Ex. 13.

15. Find the entire volume inclosed by the surface

$$\frac{x^2}{25}+\frac{y^2}{9}+\frac{z^4}{16}=1.$$

CHAPTER VI

SERIES. FURTHER FUNCTIONS

PART I. TAYLOR SERIES

§ **146. Basic Ideas.** As explained in the *Introduction*, §§ 314–17, it is often possible to find an unending series of simple algebraic terms, which will equal a given function $f(x)$.

For instance, as we shall soon verify further:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots, \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots, \tag{2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots, \tag{3}$$

$$\log (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots. \tag{4}$$

These series can be used to compute values of the functions involved; or can be substituted for the functions in troublesome calculations.

Equation (4) holds only when x is numerically less than 1.

The dots in (1)–(4) signify that each series continues indefinitely, according to its law. That is, each function is not the sum of the terms shown; but is the *limit* approached by the sum (S_n) of n such terms, as $n \to \infty$.

Any series for which S_n approaches a limit as $n \to \infty$ is said to be "convergent"; any other, "divergent." If the limit approached is equal to a function $f(x)$, we say that the series "converges to the value

of $f(x)$"; or more briefly, that it "equals $f(x)$," or that "$f(x)$ is the sum of the series." But these various phrases have only the *limit meaning* mentioned just above. (Cf. *Intro.*, § 314.)

A series for a function may proceed in powers of the variable x, as in (1)–(4), or powers of some quantity containing x, or trigonometric functions instead of powers, or otherwise. The simplest type, involving mere powers of x as in (1)–(4), is called a Maclaurin series. Let us now determine the general law for such a series.

§ **147. The General Maclaurin Series.** If a function $f(x)$ be of such a character that it can have a Maclaurin series, *i.e.*, if

$$f(x)=A+Bx+Cx^2+Dx^3+Ex^4+\cdots, \tag{5}$$

what values must the coefficients A, B, C, etc., have?

Substituting zero for x in (5) gives $f(0)=A$; thus A must be the value of the given function $f(x)$ at $x=0$. Next, differentiating (5) repeatedly, and substituting $x=0$ in each derivative (cf. *Intro.*, § 316), we find that

$$\begin{aligned}
f'(x) &= B+2Cx+3Dx^2+4Ex^3+\cdots, & f'(0) &= B,\\
f''(x) &= 2C+6Dx+12Ex^2+\cdots, & f''(0) &= 2C,\\
f'''(x) &= 6D+24Ex+\cdots, & f'''(0) &= 6D,\\
f^{\text{IV}}(x) &= 24E+\cdots, & f^{\text{IV}}(0) &= 24E.
\end{aligned}$$

$$\therefore\ B=f'(0),\quad C=\tfrac{1}{2}f''(0),\quad D=\tfrac{1}{6}f'''(0),\quad E=\tfrac{1}{24}f^{\text{IV}}(0), \tag{6}$$

and so on. The denominators 2, 6, 24, etc., are seen to be the successive factorials 2 !, 3 !, 4 !, etc.; and the reason can be seen from the manner of derivation. Thus the coefficients of the successive powers of x in (5) must be the values of the successive derivatives of $f(x)$, taken at $x=0$, and divided by the successive factorials. *I.e.*, (5) becomes

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots. \tag{7}$$

If there be any Maclaurin series for $f(x)$, this must be it.

Note the fourth term in series (7). What would the tenth term be? The n-th term?

Observe also that (7) is meaningless and non-existent if the function or any derivative becomes infinite at $x=0$.

We come now to this question: When series (7) actually exists, does it necessarily equal $f(x)$? No, not always. But it will under some conditions, as we shall now show.

§ **148. Error after n Terms.** Let E_n denote the difference between $f(x)$ and the sum S_n of the first n terms of (7). In other words, let E_n be the error if we regard a function as equal to the first n terms of its Maclaurin series.

This error depends upon the value to be used for x. It can often be estimated closely by the following

Criterion: When $x = b$, E_n cannot numerically exceed

$$\frac{G_n b^n}{n!}, \tag{8}$$

where G_n is the greatest numerical value of the n-th derivative, $f^{(n)}(x)$, anywhere from $x=0$ to $x=b$.

This criterion, the proof of which is shown in § 151, is very important, and should be remembered.

E_n is *equal* to the fraction in (8) if $x=b=0$; also if $f^{(n)}(x)$ is a constant. Otherwise E_n is *smaller* than the fraction.

If, for any given function $f(x)$, we can show the n-th derivative to be such that the fraction in (8) approaches zero as $n \to \infty$, then as E_n is no larger, we shall know that $E_n \to 0$, also. That is, $S_n \to f(x)$, or $f(x)$ is the "sum of the series," — at least for the chosen value, $x = b$.

Ex. I. Find the Maclaurin series for $\cos x$ as far as x^4. Calculate $\cos .2^{(r)}$; and test as to accuracy.

Listing the derivatives, taken at zero, we find

$$\begin{aligned} f(x) &= \cos x, & f(0) &= 1, \\ f'(x) &= -\sin x, & f'(0) &= 0, \\ f''(x) &= -\cos x, & f''(0) &= -1, \\ f'''(x) &= \sin x, & f'''(0) &= 0, \\ f^{\mathrm{IV}}(x) &= \cos x, & f^{\mathrm{IV}}(0) &= 1. \end{aligned}$$

Thus series (7) reduces to

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} \cdots, \tag{9}$$

as indicated in (1), p. 247. (Cf. also *Intro.*, §§ 316–17.)

When $x = .2$, (9) gives

$$\cos .2 = 1 - \frac{.04}{2} + \frac{.0016}{24} \cdots = .980067.$$

Although some terms have dropped out, we theoretically have here the first five terms of (7), — or, indeed, the first six, for the sixth also would be zero.

To test the accuracy of the set of terms in (9), we consider E_6, the error after six terms. By (8) this cannot exceed $G_6(.2)^6 \div 6\,!$. But as the sixth derivative is $-\cos x$, whose greatest numerical value from $x = 0$ to $x = .2$ is 1 [*i.e.*, $G_6 = 1$],

$$E_6 < \frac{(.2)^6}{6\,!}, = \frac{.000064}{720} = .000\,000\,089, \text{ approx.} \tag{10}$$

Ex. II. Show that the Maclaurin series for $\cos x$ is valid for any value, $x = b$, whatever.

Evidently G_n can never exceed 1. Hence

$$E_n < \frac{b^n}{n\,!}. \tag{11}$$

Now, as n increases indefinitely and b is fixed, n will presently far exceed b, or even $10\,b$, etc. The fraction in (11) will then decrease rapidly. For, at each step, when n increases by 1, the numerator will be multiplied simply by b while the

denominator will be multiplied by a new factor much greater than b. Thus $E_n \to 0$, regardless of what value b has.

When b is large, however, E_n may not become small until n is very large. In other words, many terms may be needed to secure fair accuracy.

EXERCISES

1. Derive the Maclaurin series for $\sin x$ as far as x^5. Calculate $\sin .5$ and test its accuracy.

2. Like Ex. 1 for the following functions, calculating each for the specified value of x:

(a) e^x, $x=.3$; (b) $\log(1+x)$, $x=.2$; (c) $(a+x)^3$, $x=2$.

3. Using (1), p. 247, as a formula, write at sight series for $\cos 2x$, $\cos .1x$, $\cos x^2$, $\cos \Delta x$.

4. Write series for $\sin 5x$, $\sin(x/2)$, $\sin \Delta x$, and $\sin(-x)$, by thinking of (2), p. 247.

5. Write by inspection series for e^{-x}, e^{2x}, e^{-x^2}. [See (3), p. 247.]

6. Find approximately the following integrals:*

(a) $\int_0^{.4} \frac{\sin x}{x}\,dx$, (b) $\int_0^{.1} \frac{e^x-1}{x}\,dx$,

(c) $\int_0^{\frac{1}{2}} e^{-x^2}\,dx$, (d) $\int_0^1 \sin x^2\,dx$.

7. What limit is approached by the fraction $(1-\cos x)/x^2$ as $x \to 0$?

8. Find the Maclaurin series for $\tan x$ as far as x^3. Compare this with the result of dividing series (2) by series (1), p. 247.

9. Find the Maclaurin series for $\sec x$ as far as x^4. Compare the result of dividing 1 by series (1).

10. Find Maclaurin's series for 10^x as far as x^4. Also substitute $x \log 10$ for x in series (3), and compare.

11. Find Maclaurin's series for $\arctan x$ as far as x^3. Also divide 1 by $(1+x^2)$ as far as x^2, integrate, and compare.

12. Parametric equations for a railway easement curve are found by integrating these expressions:

$$\frac{dx}{ds} = \cos ks^2, \qquad \frac{dy}{ds} = \sin ks^2. \tag{12}$$

Do this approximately, if $x=y=0$ when $s=0$.

13. Determine whether $\log x$ can have a Maclaurin series.

14. (a) Show series (2) valid for $x=10$. (b) Likewise series (3).

* None of these can be obtained exactly.

§ **149. Taylor Series.** For some functions a Maclaurin series is impossible. For some others such a series converges so slowly for large values of x as to be very inconvenient. Both difficulties may often be avoided by deriving a series for $f(x)$ which runs in powers of $(x-a)$, where a is some chosen constant.

To find the general expression for such a "Taylor series,"

$$f(x)=A + B(x-a) + C(x-a)^2 + D(x-a)^3 \cdots, \quad (13)$$

find successive derivatives, as in Maclaurin's case, and substitute $x=a$ in each, and in (13). Thus we find

$$A=f(a) \qquad B=f'(a), \qquad C=\frac{f''(a)}{2!}, \quad D=\frac{f'''(a)}{3!}, \cdots. \quad (14)$$

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots. \quad (15)$$

This series reduces to the Maclaurin series (7) in the special case $a=0$. But (15) is often possible for *some* value of a even if not for $a=0$.

The error E_n of (15) after n terms, when $x=b$, cannot numerically exceed

$$\frac{G_n(b-a)^n}{n!}, \quad (16)$$

where G_n is the greatest numerical value of the n-th derivative $f^{(n)}(x)$ from $x=a$ to $x=b$. (See § 151.)

Ex. I. Find the Taylor series for $\sin x$ in powers of $\left(x-\frac{\pi}{6}\right)$. Calculate $\sin .5245^{(r)}$; and test as to accuracy.

Here $f(x)= \sin x$, $f'(x)= \cos x$, $f''(x)= -\sin x$, etc.

$$\therefore f(a)=\sin\frac{\pi}{6}=.5, \quad f'(a)=\cos\frac{\pi}{6}=.86603, \quad f''(a)=-.5, \cdots.$$

Hence (15) gives as the required Taylor series:

$$\sin x=.5+.86603\left(x-\frac{\pi}{6}\right)-.25\left(x-\frac{\pi}{6}\right)^2\cdots. \quad (17)$$

Now $\frac{\pi}{6} = .5236$, nearly. Hence, when $x = .5245$, (17) gives

$$\sin .5245 = .5 + .86603(.0009) - .25(.000\,000\,81) \cdots,$$

or $\sin .5245 = .500779$, approx. Even the third term does not affect the sixth decimal place.

To apply (16), observe that G_n cannot exceed 1. Thus the error after two terms must be less than

$$\frac{1(.0009)^2}{2\,!}, \text{ or } .000\,000\,405.$$

Remark. The Maclaurin series (2) for $\sin x$ would give the same value as follows:

$$\sin .5245 = .5245 - \frac{(.5245)^3}{3\,!} + \frac{(.5245)^5}{5\,!} - \frac{(.5245)^7}{7\,!} \cdots. \tag{18}$$

But the calculations would be longer, all the terms shown in (18) being necessary for 6-place accuracy. If we had to calculate the sines of many angles near 30°, (17) would be vastly superior to (2).

§ 150. Alternative Forms. Taylor's series is often written in two other forms, as follows:

1. If we replace $(x-a)$ by h, and hence x by $a+h$, (15) will become

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2\,!}h^2 + \frac{f'''(a)}{3\,!}h^3 + \cdots. \tag{19}$$

This form is often convenient for expanding a quantity which is expressed as a function of the sum of a constant a and a variable h.

2. Since a in (19) is in reality arbitrary and can change, we may regard it as a variable x (though not the same x as above). Thus

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots. \tag{20}$$

Here h may vary, or may be held fixed. Thus (20) is convenient for a function of a sum, either of two variables or of a variable x and a constant h, — expanding in powers of h.

Remark. Since $f(x+h)-f(x)$ is the change Δy in the function while x increases by h, and h can be called Δx, we see from (20) that

$$\Delta y = f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \frac{f'''(x)}{3!}\Delta x^3 + \cdots. \tag{21}$$

The first term of this is the usual differential approximation to Δy. (Cf. § 42; also *Intro.*, § 62.) But (21) gives a closer approximation, when convergent.

EXERCISES

1. Find a Taylor series to four terms for $\sin x$, in powers of $\left(x-\frac{\pi}{4}\right)$. Calculate $\sin .786^{(r)}$ to five decimals, and test the error.

2. The same as Ex. 1 for each following case:

(*a*) $\cos x$, in powers of $(x-\pi/6)$, to find $\cos .524^{(r)}$;

(*b*) $\tan^{-1} x$, in powers of $(x-1)$, to find $\tan^{-1} 1.02$;

(*c*) e^x, in powers of $(x-2)$, to find $e^{2.065}$;

(*d*) e^x, in powers of $(x+2)$, to find $e^{-2.08}$;

(*e*) e^x, in powers of $(x+1)$, to find $e^{-.97}$.

3. The same as Ex. 1 for the following. Also, where feasible, verify the numerical results by tables:

(*a*) $\log x$, in powers of $(x-2)$, to find $\log 2.2$;

(*b*) $\log_{10} x$, in powers of $(x-3)$, to find $\log_{10} 3.05005$;

(*c*) $\sin x$, in powers of $(x-\pi/9)$, to find $\sin 21°$;

(*d*) $\sin x$, in powers of $(x-a)$, where a is the radian equivalent of 16°, to find $\sin 16°\,12'$;

(*e*) $\cos x$, in powers of $(x-a)$, where a is the radian equivalent of $27°\,35'$, to find $\cos 27°\,35'\,10''$.

4. Expand each of the following in powers of h. Note each result carefully and see whether, by factoring or collecting terms, the equation can be put into another recognizable form:

(*a*) e^{2+h}, (*b*) $\sin\left(\frac{\pi}{3}+h\right)$, (*c*) $\log(5+h)$.

5. Expand $\sin(\theta+.02)$ in powers of .02 to five terms. Check by the Addition Formula for a sine, and series (1) and (2) for $x=.02$.

6. About what change in $\log x$, while x increases from 2 to 2.105?

7. The same as Ex. 6 for e^x, as x goes from 1 to 1.08.

§ 151. Proof of the Error Criteria.

Consider first a Maclaurin series, for a special case, $n=4$. And suppose that b is positive, and that $f^{IV}(x)$ is not constant.

By definition, G_4 is a positive constant, and exceeds $f^{\mathrm{IV}}(x)$ from $x=0$ to $x=b$, — except where the two become equal. Hence, for any X between 0 and b, we have by the inequality (5), § 60:

$$\int_0^X G_4 dx > \int_0^X f^{\mathrm{IV}}(x) dx, \tag{22}$$

numerically as well as algebraically. Integrating:

$$G_4 X > f'''(X) - f'''(0). \tag{23}$$

Or, — the same thing, — we have for any x between 0 and b:

$$G_4 x > f'''(x) - f'''(0). \tag{24}$$

Since $G_4 x$ is positive, we can again apply (5), § 60, integrating from 0 to X (or x). The term $f'''(x)$ gives the difference $f''(x) - f''(0)$; but $f'''(0)$, being constant, gives only $f'''(0)x$.

$$\therefore \qquad \frac{G_4 x^2}{2} > f''(x) - f''(0) - f'''(0)x. \tag{25}$$

In fact, we can integrate repeatedly. At successive steps we have, both numerically and algebraically:

$$\frac{G_4 x^3}{3\,!} > f'(x) - f'(0) - f''(0)x - \frac{f'''(0)x^2}{2}, \tag{26}$$

$$\frac{G_4 x^4}{4\,!} > f(x) - f(0) - f'(0)x - \frac{f''(0)x^2}{2} - \frac{f'''(0)x^3}{3\,!}. \tag{27}$$

The right member is the difference between $f(x)$ and the first four terms of its Maclaurin series. That is, it is E_4. So (27) may be written

$$E_4 < \frac{G_4 x^4}{4\,!}. \tag{28}$$

When $x=b$, this takes the form (8). This establishes the criterion for the case $n=4$, $b>0$.

If b is negative, and hence also X, integrate each time from X (or x) to 0: the left member will always be positive and the argument still apply. The final right member will still be E_4.

For other values of n it is clear that similar steps will establish the criterion. (Cf. Ex. 1–3 below.)

The corresponding criterion for a Taylor series is established likewise, integrating from a to x (if $b>a$), and from x to a if $b<a$. Thus in place of equations (24) and (25), we should have when $b>a$:

$$G_4(x-a) > f'''(x)-f'''(a), \tag{29}$$

$$\frac{G_4(x-a)^2}{2} > f''(x)-f''(a)-f'''(a)\,(x-a), \tag{30}$$

and so on.

§ **152. Functions of Two Separate Variables.** These have Taylor series also. The form corresponding to (20) is *

$$f(x+h,\, y+k)$$
$$=f(x,\,y)+\left[\frac{\partial f}{\partial x}h+\frac{\partial f}{\partial y}k\right]+\frac{1}{2\,!}\left[\frac{\partial^2 f}{\partial x^2}h^2+2\frac{\partial^2 f}{\partial x\partial y}hk+\frac{\partial^2 f}{\partial y^2}k^2\right]$$
$$+\frac{1}{3\,!}\left[\frac{\partial^3 f}{\partial x^3}h^3+3\frac{\partial^3 f}{\partial x^2\partial y}h^2k+3\frac{\partial^3 f}{\partial x\partial y^2}hk^2+\frac{\partial^3 f}{\partial y^3}k^3\right]+\cdots. \tag{31}$$

That is, the change Δz in a function $z=f(x,\,y)$ is:

$$\left[\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y\right]+\frac{1}{2\,!}\left[\frac{\partial^2 z}{\partial x^2}\Delta x^2+2\frac{\partial^2 z}{\partial x\partial y}\Delta x\Delta y+\frac{\partial^2 z}{\partial y^2}\Delta y^2\right]+\cdots. \tag{32}$$

The first bracket gives the usual differential approximation.

The forms (31), (32) need not be memorized, though we shall refer to them occasionally. It may be helpful in applying them to note the similarity or analogy to the Binomial Theorem, in the higher terms.

EXERCISES

1. Prove the error criterion for Maclaurin's series in the case $n=3$. Let b be positive.

2. The same as Ex. 1 for $n=5$.

* Cf. Goursat-Hedrick: *Mathematical Analysis*, v. 1, pp. 107–108.

3. The same as Ex. 1 if b is negative.

4. Prove the error criterion for Taylor's series in the case $n=4$, with $b>a$.

5. The same as Ex. 4 for $n=2$, and $b<a$.

6. If $\int_0^b f(x)\,dx$ is approximated by integrating the first four terms of the Maclaurin series for $f(x)$, show that the resulting error cannot numerically exceed $G_4b^5/5!$. [Hint: $f(x)=S_4+E_4$; and E_4 does not exceed $G_4x^4/4!$ for any value of x from 0 to b.]

7. Like Ex. 6 when using the first 10 terms.

8. Establish a criterion like that in Ex. 6 for $\int_a^b f(x)\,dx$, and the first four terms of a Taylor series.

9. By division show that $\dfrac{1}{1-x}=1+x+x^2+x^3+\dfrac{x^4}{1-x}\cdot$ By ignoring the last fraction and integrating, obtain an expression for $\log(1-x)$. Show that its error when $x=b$ (where b lies between 0 and 1) cannot exceed $b^5/(1-b)$. [Hint: Think of a fixed value which exceeds the final fraction above until $x=b$.]

10. Like Ex. 9 when the division is continued out to x^{10} in the series.

11. Obtain by division a series for $\dfrac{1}{1+x^2}$ as far as x^8, integrate, and show that the error in the resulting expression for $\arctan x$ cannot exceed $b^{11}/(1+b^2)$, when $x=b(<1)$.

12. Expand in powers of h and k, as far as the quadratic terms:

(a) $\sin(x+h)\cos(y+k)$, (b) $e^{x+h}\sin(y+k)$, (c) $(x+h)^2(y+k)^2$.

Verify this result by multiplying out.

13. Find the Taylor series for each of the following, to four terms:

(a) $\sin x$, in powers of $(x-\pi/2)$; (b) $\log x$, in powers of $(x+5)$.

14. In finding a number y from its common logarithm x, the tables gave $y=1.3820$ when $x=.140\ 508\ 0430$; and it was desired to find y when $x=.140\ 522\ 6380$. Do this by using (21) with the relation $y=10^x$.

15. As in Ex. 14, find y when $x=.932\ 914\ 3275$, given that $y=8.5687$ when $x=.932\ 914\ 9379$.

PART II. FINITE DIFFERENCES

§ **153. Functions Defined by Tables.** A quantity (y) may vary with another (x) in a systematic way, determined by physical forces or economic factors, or otherwise. If a

definite value of y corresponds to each value of x, then y is a function of x. But our mathematical information about it may consist solely of a table of values, obtained by experiment. We then do not know how y should vary between the tabulated values. Many mathematical functions could have all those values, and yet be different between them.

If there were no possibility of further experiments to enlarge the table at pleasure, — *i.e.*, if the function were defined solely by the given table, — it would be almost wholly undefined, entirely vague, except at the given points.

It is impossible, from a table alone, to know what would be a true intermediate value of such a variable. But if some *hypothesis* be adopted, as to how y should vary, intermediate values can then be calculated *on that basis*. And, by comparing those values with further experimental observations, the reasonableness of the hypothesis can be tested.

It is customary to assume that y will vary continuously and gradually, rather than by jumps or jerks. Thus we draw a smooth graph, and read off intermediate values from it. Or we interpolate by proportional parts, ignoring the curvature of the graph for a short distance. These methods, however, are too inaccurate in many cases.

A Taylor series is not readily obtainable from a mere table; but there is another series somewhat analogous to Taylor's which can be used with many tables. (§ 154.)

Sometimes we can discover a formula for a given table, by the methods shown in the *Introduction*, §§ 32, 175, 323–24. And we usually assume that such a formula will be correct for intermediate values also. For a small number of interpolations, however, the new method will be quicker, when applicable.

§ **154. Interpolation by Successive Differences.** Let the values of x in a given table run at constant intervals Δx. Form the differences Δy between consecutive y values; then between consecutive Δy's; and so on. (The Δy's are called

"first differences"; their differences are called "second differences," and denoted by $\Delta^2 y$, etc.) In subtracting we take each value from the one following, *i.e.*, below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$y_1 - y_0$		
x_1	y_1		$y_2 - 2y_1 + y_0$	
		$y_2 - y_1$		$y_3 - 3y_2 + 3y_1 - y_0$
x_2	y_2		$y_3 - 2y_2 + y_1$	
		$y_3 - y_2$		
x_3	y_3			

Observe how the successive Δ's run, in the top sloping row, running downward from y_0:

$$\Delta y = y_1 - y_0, \quad \Delta^2 y = y_2 - 2y_1 + y_0, \quad \Delta^3 y = y_3 - 3y_2 + 3y_1 - y_0, \cdots. \tag{33}$$

Solving these for y_1, y_2, y_3, etc., in terms of the Δ's gives

$$y_1 = y_0 + \Delta y, \quad y_2 = y_0 + 2\Delta y + \Delta^2 y, \quad y_3 = y_0 + 3\Delta y + 3\Delta^2 y + \Delta^3 y \cdots. \tag{34}$$

Note the coefficients of the Binomial Theorem. The same rule holds at the next step (Ex. 10, p. 262):

$$y_4 = y_0 + 4\Delta y + 6\Delta^2 y + 4\Delta^3 y + \Delta^4 y. \tag{35}$$

In general, for every integral value of n in the range of the table, the following formula holds: *

$$y_n = y_0 + n\Delta y + \frac{n(n-1)}{2!}\Delta^2 y + \frac{n(n-1)(n-2)}{3!}\Delta^3 y + \cdots. \tag{36}$$

If we let $y = f(x)$, and note that $x_n = x_0 + n\Delta x$, then we may write $y_n = f(x_n) = f(x_0 + n\Delta x)$. Thus (36) becomes

$$f(x_0 + n\Delta x) = f(x_0) + n\Delta f + \frac{n(n-1)}{2!}\Delta^2 f \cdots. \tag{37}$$

For integral values of n, (37) gives the tabulated values of $f(x)$ exactly. For fractional values of n, it usually gives intermediate values of $f(x)$ closely. In fact, if the table is

* This may be verified in any given case by direct substitution. For a general proof of the conclusions stated here, see Whittaker and Robinson: *The Calculus of Observations*, pp. 1–12, 364.

very extensive, so that numerous successive Δ's are available, and if $f(x)$ has a valid Taylor series in the interval covered by the table, then (37) converges to the true value of the function as $n \to \infty$. Usually, the Δ's decrease rapidly, and presently become negligible, so that a few terms of (37) suffice.

If the Δ's are constant at any step, an exact formula for the table can also be found as in *Intro.*, § 324.

Any pair of values in the table may be regarded as (x_0, y_0), provided we take the successive Δ's which start from that pair. Also, we may regard the table as starting at the bottom rather than at the top, provided we form differences accordingly. The successive Δ's for any pair will then run in a line sloping upward.

To interpolate inversely, *i.e.*, to find x when y is given, we may proceed as in Ex. II below.

Ex. I. Given the first two columns of the following table, find sin 76° 35′.

$x°$	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$
75	.96593			
		.00437		
76	**.97030**		$-.00030$	
		.00407		$+.00001$
77	.97437		$-$**.00029**	
		.00378		$-$**.00001**
78	.97815		$-.00030$	
		.00348		
79	.98163			

The tabular interval Δx is 1°. Take 76° as x_0 and 76° 35′ as $x_0 + n\Delta x$. The fraction n must then be $\frac{35}{60}$ or $\frac{7}{12}$. Thus (37) gives

$$\sin 76°\, 35' = .97030 + \tfrac{7}{12}(.00407) + \frac{\frac{7}{12}\left(-\frac{5}{12}\right)}{2}(-.00029) + \cdots.$$

That is, sin 76° 35′ = .97271, correct to five figures.

Remarks. (I) Using Proportional Parts, the last term here would be omitted, with a resulting error of .00004.

(II) If we needed to figure from the bottom of the table upward, we would take $\Delta x = -1°$, $x_0 = 77°$, $x_0 + n\Delta x = 76° 35'$, so that $n = \frac{25}{60} = \frac{5}{12}$. And, then, $\Delta y = -.00407$, $\Delta^2 y = -.00030$, etc.

$$\therefore \sin 76° 35' = .97437 + \tfrac{5}{12}(-.00407) + \frac{\frac{5}{12}(-\frac{7}{12})}{2}(-.00030) + \cdots. \qquad (38)$$

This again gives $\sin 76° 35' = .97271$.

Ex. II. From the above table find arcsin .97318.

Evidently this lies between 76° and 77°, but the fraction n is unknown. Starting from 76°, by (37):

$$.97318 = .97030 + n(.00407) + \frac{n(n-1)}{2}(-.00029) + \cdots. \qquad (39)$$

Temporarily ignoring the second difference, as in Proportional Parts, we approximate n roughly:

$$.97318 = .97030 + n(.00407). \qquad \therefore\ n = \frac{.00288}{.00407} = .707.$$

Although this is not exact, it can be used to get a very accurate value for the small last term in (39), viz.:

$$\frac{.707(-.293)}{2}(-.00029), \qquad \text{or } +.00003.$$

Substituting this for the last term in (39), transposing and again solving for n, more accurately: $n = .700.^{+}$

$$\therefore \text{ arcsin } .97318 = 76° + .7(60') = 76° 42'.$$

EXERCISES

1. From the table in Ex. I above find sin 76° 40′. Check the result by consulting a larger table.

2. (*a*) The same as Ex. I for sin 76° 24′. (*b*) Likewise for arcsin .97169.

3. From Table I below find $e^{1.23}$. Check by logarithms.

4. Table II gives the present value of a certain annuity for several interest rates r. Find P for $r = .058$.

5. Table III gives the number of survivors of a group at various ages A yr. Find N at $A=72.3$.

6. Table IV gives the distance of Halley's Comet from the sun T yr. since it was nearest (April, 1910). Approximate r in April, 1920.

7. Table V gives the right ascension and declination of the moon at several hours of a certain day. (*a*) Find R at $2^h 18^m$. (*b*) Find D at $1^h 48^m$.

8. From Table VI find $\sqrt{5.05}$.

9. From Table VII find $\sqrt{2730}$.

TABLE I

x	e^x
1.2	3.320
1.3	3.669
1.4	4.055
1.5	4.482
1.6	4.953

TABLE II

r	P
.02	16.3514
.03	14.8775
.04	13.5903
.05	12.4622
.06	11.4699

TABLE III

A	N
71	36178
72	33730
73	31243
74	28738
75	26237

TABLE IV

T	r
8	19.0
18	28.9
28	33.8
38	35.4
48	33.8

TABLE V

t	R			D		
h	h	m	s	°	′	″
1	14	31	33.04	10	1	51.3
2	14	33	53.98	10	11	54.5
3	14	36	15.16	10	21	54.3
4	14	38	36.58	10	31	50.7
5	14	40	58.26	10	41	43.5

TABLE VI

N	$\sqrt{N}$
5.0	2.23607
5.1	2.25832
5.2	2.28035
5.3	2.30217
5.4	2.32379

TABLE VII

N	N^2
51	2601
52	2704
53	2809
54	2916
55	3025

10. Verify equation (35) for y_4. [In the general first table of § 154, obtain the leading $\Delta^4 y$. Substitute it, together with the expressions for Δy, $\Delta^2 y$, $\Delta^3 y$ in (35).]

11. In Table VIII one value is incorrect. Discover which; and find its true value by adjusting the Δ's. As a check, calculate the value from log 3.100 by (37).

12. The same as Ex. 11 for Table IX.

TABLE VIII

N	$\log N$
3.100	.49136
3.101	150
3.102	164
3.103	178
3.104	192
3.105	216
3.106	220
3.107	234

TABLE IX

N	$1/N$
4.70	.212 766
4.71	314
4.72	.211 864
4.73	416
4.74	.210 970
4.75	526
4.76	094
4.77	.209 644

PART III. HYPERBOLIC FUNCTIONS

We proceed next to define a very important class of functions, but first make one or two more observations of a general character.

§ 155. Functions Defined by Series. Many higher mathematical functions are defined and studied by means of infinite series, so chosen as to facilitate the solution of some important problem. In the present course we have time to mention only one instance, viz. imaginary powers of e.

As noted in the *Introduction*, § 320, the Maclaurin series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots, \tag{40}$$

which holds for every real value of z, is taken as *defining* what shall be understood by e^z when z is imaginary.

In particular, if $z = iy$, where y is real and $i = \sqrt{-1}$, then (40) gives, upon simplifying:

$$e^{iy} = 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} \cdots. \tag{41}$$

Separating real terms from imaginaries:

$$e^{iy} = \left[1 - \frac{y^2}{2!} + \frac{y^4}{4!} \cdots\right] + i\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} \cdots\right]. \tag{42}$$

Comparing with (1) and (2), p. 247, we get Euler's very important formula,

$$\boldsymbol{e^{iy} = \cos y + i \sin y.} \tag{43}$$

That is, in the notation of the *Introduction*, § 349,

$$e^{iy} = \text{cis } y. \tag{44}$$

If z is not a pure imaginary iy, but a complex number $x + iy$, then (40) assumes a more complicated form. But it can be shown (cf. Ex. 3, p. 266) that this same form would

result from multiplying together the two Maclaurin series for e^x and e^{iy}. Thus

$$e^{x+iy}=e^x\cdot e^{iy}. \tag{45}$$

That is, in multiplying such powers, exponents may be added as formerly. The other familiar laws of exponents also hold. (Cf. Ex. 4, p. 266.)

Equation (45) may also be written in the form

$$e^{x+iy}=e^x[\cos y+i\sin y]. \tag{46}$$

By means of this we can change imaginary powers of e into real exponential and trigonometric functions, with the imaginary element i occurring merely as a coefficient in one term. This will be exceedingly useful in Chapter VIII.

§ **156. More Exponential-Trigonometric Relations.** From the basic equation (43), other relations between the exponential and trigonometric functions follow. Replacing y by $-y$ in (43), and combining the resulting equation with (43), gives (as in Ex. 5, p. 267):

$$\cos y=\frac{e^{iy}+e^{-iy}}{2}, \qquad \sin y=\frac{e^{iy}-e^{-iy}}{2i}. \tag{47}$$

These equations are useful in certain reductions, especially in Mathematical Astronomy, where it is often necessary to change a power like $\cos^n\theta$ into a linear form, in terms of functions of multiple angles.

To illustrate in the simple case of $\cos^3\theta$, we find from (47):

$$\cos^3\theta=\left[\frac{e^{i\theta}+e^{-i\theta}}{2}\right]^3=\tfrac{1}{8}[e^{3i\theta}+3\,e^{2i\theta}e^{-i\theta}+3\,e^{i\theta}e^{-2i\theta}+e^{-3i\theta}].$$

Combining exponents and collecting corresponding terms:

$$\cos^3\theta=\tfrac{1}{8}[(e^{3i\theta}+e^{-3i\theta})+3(e^{i\theta}+e^{-i\theta})].$$

But the first parenthesis, by (47), is $2\cos 3\theta$; and the second, $2\cos\theta$.

$$\therefore \qquad \cos^3\theta=\tfrac{1}{4}[\cos 3\theta+3\cos\theta]. \tag{48}$$

In like manner, by using the Binomial Theorem, general formulas can be obtained for $\cos^n\theta$ and $\sin^n\theta$.

§ 157. Functions of Sectorial Areas. By way of introduction to the new functions shortly to be studied, let us look at the ordinary trigonometric functions in another light.

In a circle of unit radius (Fig. 96), let A be the area of a sector between two radii whose included angle is $2\,\theta$. Then $A = \frac{1}{2}(r^2)(2\,\theta) = \theta$. But the half-chord y and its distance x from the center are

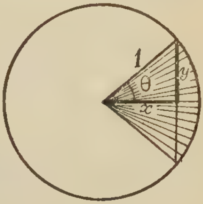


FIG. 96.

$$x = r\cos\theta = \cos\theta, \qquad y = r\sin\theta = \sin\theta.$$

Replacing θ by the equal value A:

$$x = \cos A, \qquad y = \sin A. \tag{49}$$

Without thinking further of angle θ: if we let the area A vary, x and y must vary with it, as shown by (49). Thus the lines x and y are trigonometric functions, — also called *circular functions,* — of the sectorial area A.

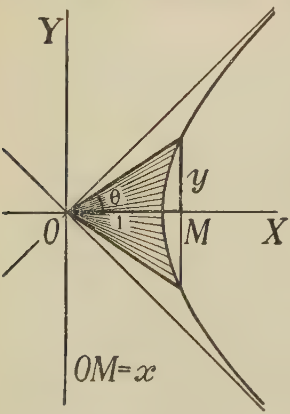


FIG. 97.

In a trigonometric table, with radian measure, if we regard the angle column as showing values of the area A, the cosine and sine columns will show values of the lines x and y.

Consider now the corresponding area A and lines x and y, for the unit rectangular hyperbola

$$x^2 - y^2 = 1, \quad \text{or} \quad y^2 = x^2 - 1. \tag{50}$$

By analogy we shall call x and y *hyperbolic* functions of the sectorial area A (Fig. 97): x the hyperbolic cosine, and y the hyperbolic sine, written

$$x = \cosh A, \qquad y = \sinh A. \tag{51}$$

Other notations, often seen, are ch$A$ and sh$A$. But those given here are the most generally used. They may be read briefly "cose-h," "sine-h."

These new functions are expressible in terms of ordinary exponentials. For A is the difference between a triangular area $(=xy)$ and twice the "area S under the curve" from $x=1$ to $x=x$. But

$$2\,S=2\int_1^x y\,dx=2\int_1^x \sqrt{x^2-1}\,dx=x\sqrt{x^2-1}-\log(x+\sqrt{x^2-1}).$$

As $\sqrt{x^2-1}=y$, this may be written $2\,S=xy-\log\,(x+y)$.

$$\therefore \qquad A=xy-2\,S=\log\,(x+y). \qquad (52)$$

$$\therefore \qquad x+y=e^A. \qquad (53)$$

Dividing this into $x^2-y^2=1$ gives further

$$x-y=e^{-A}. \qquad (54)$$

Combining (53) and (54) gives, for x and y, the hyperbolic cosine and sine:

$$x=\cosh A=\frac{e^A+e^{-A}}{2}, \qquad y=\sinh A=\frac{e^A-e^{-A}}{2}. \qquad (55)$$

EXERCISES

1. Plot $x^2-y^2=1$ carefully, taking $x=1$, 1.2, 1.4, etc., to $x=2$. Measure the sectorial area A, of Fig. 97, when $x=2$. Using that value of A, calculate x and y from (55). Check.

2. What would you suggest as a suitable quantity to call the hyperbolic tangent of A in Fig. 97? What would be its value in terms of exponentials?

3. Multiply the Maclaurin series for e^u by that for e^v, keeping product terms as far as the third degree. Compare with the Maclaurin series for e^{u+v} as far as $(u+v)^3$.

4. Reduce each of the following to a standard "cis" form (*Intro.*, §§ 350–51):

$$\text{cis}\,\theta\cdot\text{cis}\,\phi, \quad \text{cis}\,\theta\div\text{cis}\,\phi, \quad (\text{cis}\,\theta)^7, \quad \sqrt[3]{\text{cis}\,\phi}.$$

Express each of these reductions in exponential form. What laws of exponents are illustrated?

5. Show that

$$e^{-iy} = \cos y - i \sin y. \tag{56}$$

Also derive (47), p. 264, more in detail than was done in § 156.

6. Compare (55) with (47). What differences in form?

7. Express in terms of real exponential and trigonometric functions:

(*a*) e^{A+iB}, $e^{\theta+i\phi}$, $e^{5\theta-i\phi}$, $e^{(2-i)t}$;

(*b*) $e^{(3+2i)t}$, $e^{(7+\sqrt{3}i)x}$, $e^{(6-9i)\phi}$, $e^{(8-\sqrt{2}i)s}$.

8. Show that the following reduce to real forms entirely:

(*a*) $e^{(2+i)x}+e^{(2-i)x}$, (*b*) $(7+3\,i)e^{(4+2i)t}+(7-3\,i)e^{(4-2i)t}$,

(*c*) $ie^{\sqrt{3}ix}-ie^{-\sqrt{3}ix}$, (*d*) $\dfrac{5+2\,i}{4}e^{(1+i)t}+\dfrac{5-2\,i}{4}e^{(1-i)t}$.

9. Reduce $\sin^3\theta$ to a linear form, in functions of multiple angles.

10. The same as Ex. 9 for each of the following:

(*a*) $\cos^4\theta$, (*b*) $\sin^4\theta$, (*c*) $\cos^5\theta$.

11. By (47) show that $(1-2\,r\cos\phi+r^2)^n$ can be factored into $(1-re^{i\phi})^n\cdot(1-re^{-i\phi})^n$. [This step is very useful in Mathematical Astronomy.]

12. From Table I find (*a*) $\tan 27° 30'$; (*b*) $\tan^{-1} .4877$.

13. Table II gives the vapor tension of water (t mm.) at various temperatures ($T°$). Find t when $T = 90.6$.

TABLE I

θ	$\tan\theta$
°	
25	.4663
30	.5774
35	.7002
40	.8391
45	⋯ ?

TABLE II

T	t
90	525.47
91	545.77
92	566.71
93	588.33
94	610.64

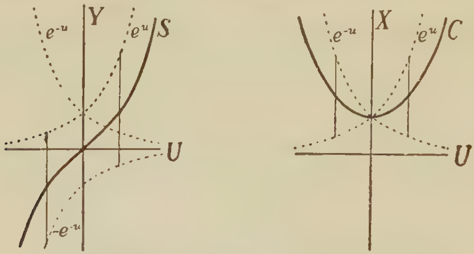
§ 158. Hyperbolic Functions. The hyperbolic sine and cosine, mentioned briefly in § 157, are only two of a set of "hyperbolic functions," which are highly important in electrical engineering, mechanics, cartography, and elsewhere.

We shall now adopt complete definitions, analytic in form, which will be consistent with the geometrical definitions of § 157 and convenient for later use.

By the hyperbolic sine and cosine of any variable u, we shall understand:

$$\sinh u = \frac{e^u - e^{-u}}{2}, \qquad \cosh u = \frac{e^u + e^{-u}}{2}. \tag{57}$$

The graphs of these functions are shown in Fig. 98, — together with those of e^u and e^{-u} for comparison.



(a) $y = \sinh u$. (b) $x = \cosh u$.

FIG. 98.

Remarks. (I) There is just one value of y or x for any given value of u. But, for a given value of x there may be no value, or two values, of u, — and hence also of y.

(II) In any vertical line, the curve C is midway between the graphs of e^u and e^{-u}. And S is midway between the graph of e^u and the reflection of the graph of e^{-u} in the base line.

The hyperbolic tangent, cotangent, secant, and cosecant we define in terms of sinh u and cosh u, by relations strictly analogous to those which hold for the trigonometric functions:

$$\tanh u = \frac{\sinh u}{\cosh u},$$
$$\operatorname{ctnh} u = \frac{1}{\tanh u}, \qquad \operatorname{sech} u = \frac{1}{\cosh u}, \qquad \operatorname{csch} u = \frac{1}{\sinh u}. \tag{58}$$

Tables are commonly printed for the first three functions, the others being calculated as reciprocals, when needed. A small table is given on p. 500; for accurate interpolation, successive differences should be used, as in § 154.

§ **159. Relations.** Besides the defining relations (58) among the hyperbolic functions, many other relations hold, analogous to, though often not the same as, the various relations among the trigonometric functions.

Squaring $\cosh u$ and $\sinh u$ in (57), and subtracting, we get at once:

$$\cosh^2 u - \sinh^2 u = 1. \tag{59}$$

Also, dividing (59) by $\cosh^2 u$ and comparing (58):

$$1 - \tanh^2 u = \operatorname{sech}^2 u. \tag{60}$$

Similarly we find, as in Ex. 3 below:

$$\operatorname{ctnh}^2 u - 1 = \operatorname{csch}^2 u. \tag{61}$$

Note how (59)–(61) differ from the analogous trigonometric formulas. The identity (59) should be memorized, as many other relations follow directly from it.

Among the numerous hyperbolic formulas are some for functions of a double variable $2u$, some for the functions of the sum of two variables $(u + v)$, some for functions of imaginary variables, etc. These are covered fully in special texts on the subject.

§ 160. Derivatives and Integrals. Differentiating the value of $\sinh u$ in (57) gives that of $\cosh u$; and vice versa. Hence

$$d(\sinh u) = \cosh u\, du, \qquad d(\cosh u) = \sinh u\, du. \tag{62}$$

Hence, inversely: Integrating either $\sinh u$ or $\cosh u$ gives the other, plus a possible constant. (Note the absence of minus signs.)

Again, differentiating the several equations (58), and simplifying, will give standard differentiation formulas for $\tanh u$, $\operatorname{ctnh} u$, $\operatorname{sech} u$, and $\operatorname{csch} u$. (Cf. p. 490.)

The integration of various expressions involving hyperbolic functions closely parallels the procedure for the analogous trigonometric expressions.

E.g., to integrate $\cosh^3 u\, du$, split off $\cosh u\, du$ as the differential of $\sinh u$; and change the remaining $\cosh^2 u$ to $1 + \sinh^2 u$, by (59). Thus the integrand takes the form $(1 + v^2)dv$, where v denotes $\sinh u$.

EXERCISES

1. Express tanh u in terms of exponential functions.

2. Show that $e^u = \cosh u + \sinh u$. Contrast this with formula (43) for trigonometric functions.

3. Derive (59) algebraically. Also obtain it from Fig. 97. From (59) derive (61).

4. Look up the values of sinh .8, cosh .8, tanh .8. Verify that they fulfill the relations (58) and (59). Find sech .8. Also calculate sinh .8 from $e^{.8}$ and $e^{-.8}$, and check.

5. Find sinh .52; also tanh .54.

6. For a telegraph wire of length L km., supplied with an e. m. f. of 200 volts at one end and having the other end grounded, the currents entering the wire initially and leaving it finally are

$$I = \frac{200}{r \tanh \theta}, \quad I' = \frac{200}{r \sinh \theta},$$

where r is the surge resistance, and, for a certain uniform leakance, $\theta = .005\,L$. Find I and I' if $r = 4000$ and $L = 300$.*

7. The same as Ex. 6 if $\theta = .0025\,L$, $r = 4200$, and $L = 1000$.

8. Derive formulas (62) in detail. Also differentiate at sight:

$$\sinh 2x, \qquad \cosh 5x, \qquad 6 \sinh .2\,\phi, \qquad 4 \cosh (t/8).$$

9. The height (y ft.) of a certain cable above level ground, at a horizontal distance of x ft. from the middle, is $y = 100 \cosh .01\,x - 70$. How high is it at the middle? At one end A, where $x = 50$? What is the slope at A?

10. Derive a differentiation formula for tanh u:

(a) From (58); (b) From the exponential value of tanh u.

11. Integrate at sight:

$$\sinh 4\theta\, d\theta, \qquad \cosh 7x\, dx, \qquad 10 \sinh 6t\, dt, \qquad 9 \cosh \frac{x}{4}\, dx,$$

$$\sinh \frac{x}{a}\, dx, \qquad \cosh .2\,x\, dx, \qquad c \sinh \frac{\phi}{c}\, d\phi, \qquad k \cosh \frac{\theta}{3}\, d\theta.$$

12. Show that $\cosh^2 u = \frac{1}{2}(1 + \cosh 2u)$. Also derive an analogous formula for $\sinh^2 u$.

13. Find $\int \cosh^2 u\, du$ by using the formula in Ex. 12; also by using the exponential value for cosh u.

* Cf. A. E. Kennelly, *The Application of Hyperbolic Functions to Electrical Engineering Problems*, Chap. II.

14. Find the following integrals:

(*a*) $\int \sinh^3 u\, du$, (*b*) $\int \cosh^5 x\, dx$, (*c*) $\int \sqrt{1+\sinh^2 u}\, du$,

(*d*) $\int u \cosh u\, du$, (*e*) $\int u \sinh u\, du$, (*f*) $\int \sqrt{\cosh^2 u - 1}\, du$.

15. Find the length of the curve $y=3 \cosh (x/3)$ from $x=0$ to $x=9$.

16. Find the entire length of the cable in Ex. 9.

§ **161. Inverse Functions.** If $x=\cosh u$, then u is called the "inverse hyperbolic cosine of x," written

$$u=\cosh^{-1} x. \tag{63}$$

This can also be expressed in a familiar logarithmic form. For, by definition of $\cosh u$,

$$x=\frac{e^u+e^{-u}}{2}. \tag{64}$$

Letting $e^u=v$ (whence $e^{-u}=1/v$), and solving for v we find

$$2\,x=v+\frac{1}{v}, \qquad \therefore\ v^2-2\,xv+1=0.$$

And this quadratic gives for v or e^u:

$$v=e^u=x\pm\sqrt{x^2-1}. \tag{65}$$

$$\therefore \qquad \boldsymbol{\cosh^{-1} x=u=\log (x\pm\sqrt{x^2-1})}. \tag{66}$$

The double sign in (66) agrees with the fact that, for a given x, there are two values of u. (Cf. Fig. 98.) The principal or positive value takes the upper sign.

Unless the contrary is stated, we shall hereafter use only the principal value.

Similarly, if $x=\sinh u$, we write $u=\sinh^{-1} x$. And this inverse hyperbolic sine can be expressed as

$$\sinh^{-1} x=u=\log (x+\sqrt{x^2+1}). \tag{67}$$

(Why is there no ambiguity of sign here?)

Derivatives. By either method in Ex. 11, p. 274, we find

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{x^2+1}}, \tag{68}$$

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}. \tag{69}$$

Reversing these differentiations, and then replacing x by x/a on both sides, we may rewrite two earlier integration formulas as follows:

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C, \tag{70}$$

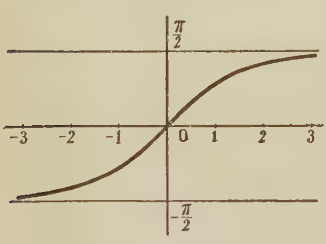
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C. \tag{71}$$

These new formulas need not be memorized, but should be familiar. They are useful chiefly in connection with "differential equations," discussed in Chapter VIII.

§ 162. The Gudermannian. A function sufficiently important to mention, even in this brief treatment of the hyperbolic functions, is defined thus:

The "Gudermannian of x," written $\operatorname{gd} x$, is the angle whose tangent is the hyperbolic sine of x:

$$\phi = \operatorname{gd} x \qquad \text{means} \qquad \phi = \tan^{-1}(\sinh x). \tag{72}$$



$y = \operatorname{gd} x = \tan^{-1}(\sinh x)$.

FIG. 99.

Values of this function are tabulated on p. 500, — in radians as here defined, and with the equivalent in degrees. The graph runs as in Fig. 99.

From (72) we can also get x in terms of ϕ, as the "inverse Gudermannian." We find $\sinh x = \tan\phi$, or $x = \sinh^{-1}(\tan\phi)$.

Hence, by (67):

$$x = \log(\tan\phi + \sqrt{\tan^2\phi + 1}). \tag{73}$$

But, by (72), $x = \text{gd}^{-1}\phi$.

$$\therefore \qquad \text{gd}^{-1}\phi = \log(\sec\phi + \tan\phi). \tag{74}$$

Since the right member of (74) is the well-known value of $\int \sec\phi\, d\phi$, we may rewrite that integral thus:

$$\int \sec\phi\, d\phi = \text{gd}^{-1}\phi. \tag{75}$$

It can also be proved that

$$\int \text{sech}\, x\, dx = \text{gd}\, x. \tag{76}$$

These various equations relating to the Gudermannian need not be memorized.

EXERCISES

1. Look up gd 1. Check roughly by looking up first sinh 1, and then the angle whose tangent is the latter value. (Do not bother to interpolate.)

2. Look up gd^{-1} .7985. Check by putting $\phi = .7985$ (or $45° 45'$) in (74), and looking up the logarithm there present.

3. Look up gd 1.2; also gd^{-1} 1.2.

4. If a ship constantly sails exactly northeast, its latitude $L^{(r)}$ will vary thus with its longitude $\theta^{(r)}$: $L = \text{gd}(\theta + C)$. If $L = .4$ when $\theta = .5$, find L when $\theta = .8$; also when $\theta = 1.1$.

5. Under certain conditions the voltage e and the current i at a point P of a telegraph wire, l km. from the initial end, are

$$e = 200 \cosh .005\, l - 220 \sinh .005\, l,$$
$$i = .055 \cosh .005\, l - .050 \sinh .005\, l.$$

Find e and i when $l = 100$. Also when $l = 200$.

6. Find the area under the curve $y = 5 \cosh .2\, x$, from $x = 0$ to $x = 10$.

7. A wire of constant linear density, $D = k$, hangs in the curve $y = 10 \cosh .1\, x$. Find $\bar{y}$ for the centroid of the portion between $x = -5$ and $x = 5$.

8. Find the slope and inclination of the wire in Ex. 7 at $x = 5$. Also show that the inclination at any point is gd $.1\, x$.

9. (*a*) What curve has the parametric equations $x = 4 \cosh\phi$, $y = 3 \sinh\phi$? (*b*) Find parametric equations for the general hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, in terms of hyperbolic functions.

10. Derive equation (67).

11. If $u = \cosh^{-1} x$, find du/dx from (66). Also find it as follows: Write $x = \cosh u$, find dx/du, and invert. To simplify, use (59) to show that $\sinh u = \sqrt{x^2 - 1}$.

12. Like Ex. 11, if $u = \sinh^{-1} x$.

PART IV. FOURIER SERIES

§ 163. General Idea. For many purposes a Maclaurin or Taylor series, which proceeds in algebraic powers, is less convenient than a trigonometric series of a certain type. This latter, devised by Fourier, involves sines or cosines of multiple angles; viz.

$$f(x) = A_0 + A_1 \cos x + A_2 \cos 2x + A_3 \cos 3x + \cdots, \quad (77)$$

or

$$f(x) = B_0 + B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + \cdots, \quad (78)$$

or a combination of these. All the terms are linear. A rather long calculation shows that the term B_0 is superfluous. We henceforth omit it.

A proof of convergence is outside the scope of this text. But the facts are these:*

(I) If series (77) or (78) is to equal $f(x)$ for all values from $x = 0$ to $x = \pi$, correct coefficients (A's or B's) can readily be found by the method below.

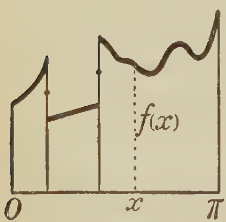


FIG. 100.

(II) Conversely, (77) or (78) with the coefficients so determined will actually converge to the value of $f(x)$, from $x = 0$ to $x = \pi$, — provided $f(x)$ has at every point in this interval a definite finite value, and provided it has only a limited number of maxima and minima, or of finite discontinuities, if any. (Fig. 100.)

Similar statements hold for wider intervals than from 0 to π. But the determination of the coefficients is then different.

* Cf. Goursat-Hedrick, *Mathematical Analysis*, v. 1, p. 414 ff.

§ 164. **Fourier Coefficients: Interval 0 to π.** Let us now find a formula for the coefficients A_0, A_1, A_2, etc., in (77), if that cosine series is to be valid from 0 to π. We shall assume that such a series can be integrated term by term, as in a finite sum. The exact conditions under which this is legitimate are discussed in more advanced courses.*

Integrating both sides of (77) from 0 to π gives

$$\int_0^\pi f(x)dx = \int_0^\pi A_0 dx + \int_0^\pi A_1 \cos x\, dx + \int_0^\pi A_2 \cos 2x\, dx + \cdots. \quad (79)$$

On the right side every integral after the first vanishes, since the sine of zero or of any multiple of π is zero. Thus (79) reduces to

$$\int_0^\pi f(x)dx = A_0\pi, \qquad \therefore\ A_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx. \quad (80)$$

This furnishes a formula or rule for finding A_0 in all such cases, for any given function $f(x)$.

To find a similar formula for any other A, say A_n, multiply both sides of (77) by $\cos nx$, and then integrate:

$$\int_0^\pi f(x)\cos nx\,dx = \int_0^\pi A_0 \cos nx\, dx + \int_0^\pi A_1 \cos x \cos nx\, dx + \cdots. \quad (81)$$

The first integral on the right vanishes. So does the second if $n \neq 1$. For by formula (79), p. 497, this integral is

$$A_1\left[\frac{\sin (n+1)x}{2(n+1)} + \frac{\sin (n-1)x}{2(n-1)}\right]_0^\pi, \quad (82)$$

and both sines become zero at each limit. Similarly for any other integral in (81) involving A_k, if $k \neq n$. Thus *all the*

* Cf. T. J. I'a Bromwich, *An Introduction to the Theory of Infinite Series*, p. 115 ff.

integrals on the right side of (81) *disappear, except the one involving* A_n. The latter is

$$\int_0^\pi A_n \cos nx \cos nx\, dx, \quad \text{or } A_n \int_0^\pi \cos^2 nx\, dx. \tag{83}$$

This last integral has the value $\frac{1}{2}\left[x + \frac{1}{2\,n}\sin 2\,nx\right]_0^\pi$, or $\frac{\pi}{2}$. Hence (81) reduces to

$$\int_0^\pi f(x) \cos nx\, dx = A_n \left(\frac{\pi}{2}\right). \tag{84}$$

We therefore have for all the coefficients in (77):

$$A_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\, dx, \tag{85}$$

except when $n=0$, for which case we have

$$A_0 = \frac{1}{\pi}\int_0^\pi f(x)\, dx. \tag{86}$$

A formula similar to (85) holds for the coefficients B_n in the sine series (78), with B_0 superfluous:

$$B_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\, dx. \tag{87}$$

For any given function $f(x)$ we use (85)–(87), as needed, discover how the coefficients run numerically, and then write as many terms of (77) or (78) as are desired. The case of a combination series will be mentioned presently. (§ 165.)

Ex. I. Find the Fourier sine series for e^x, valid from $x=0$ to $x=\pi$.

Here (87) gives, by using formula (88), p. 498,

$$B_n = \frac{2}{\pi}\int_0^\pi e^x \sin nx\, dx = \frac{2}{\pi}\left[e^x\, \frac{\sin nx - n \cos nx}{1+n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{e^\pi n \cos n\pi}{1+n^2} + \frac{n}{1+n^2}\right] = \frac{2\,n}{\pi(1+n^2)}[1 - e^\pi \cos n\pi].$$

Now, for odd values of n, $\cos n\pi = -1$; but, for even values, $\cos n\pi = 1$. *E.g.*, $n=1$ and $n=2$ give

$$B_1 = \frac{2(1)}{\pi(2)}[1-e^{\pi}(-1)] = \frac{1}{\pi}[1+e^{\pi}],$$

$$B_2 = \frac{2(2)}{\pi(5)}[1-e^{\pi}(1)] = \frac{4}{5\,\pi}[1-e^{\pi}].$$

Similarly for $n=3$, 4, etc., we find

$$B_3 = \frac{3}{5\,\pi}[1+e^{\pi}], \qquad B_4 = \frac{8}{17\,\pi}[1-e^{\pi}], \text{ etc.}$$

Substituting these in (78), — and calling $1+e^{\pi}=S$, and $1-e^{\pi}=D$, for brevity, — the required series is

$$e^{x} = \frac{1}{\pi}\{S \sin x + \tfrac{4}{5}\, D \sin 2\,x + \tfrac{3}{5}\, S \sin 3\,x + \tfrac{8}{17}\, D \sin 4\,x \cdots\}. \quad (88)$$

Remarks. (I) Strictly speaking, series (88) does not equal e^{x} at $x=0$ or $x=\pi$. (Why not, evidently?) But it equals e^{x} for any value of x slightly larger than zero, however near. And the limit approached by the two as $x \to 0$ is the same. Similarly near π.

(II) Series (88) cannot equal e^{x} outside the interval 0 to π. For instance, at $x=-\pi/2$, every sine term has the negative of its value at $x=\pi/2$; but $e^{-\frac{\pi}{2}}$ is not the negative of $e^{\frac{\pi}{2}}$. Somewhat similar arguments apply elsewhere.

(III) The chief uses of Fourier series lie outside the scope of this text, — in the study of wave motions (of heat, sound, etc.).

EXERCISES

For each function in Ex. 1–5, find a Fourier series, valid from 0 *to* π.

1. For the number 1; that is, $f(x)=1$ in (85)–(87):

(a) The sine series, (b) The cosine series.

Observe how quickly the cosine series ends in this case.

2. For x itself; that is, $f(x)=x$:

(a) The sine series, (b) The cosine series.

3. For $f(x)=11-5\,x$:

(a) The sine series, (b) The cosine series.

Compare each result with the corresponding series in Ex. 1–2.

4. For $1+t$, the sine series in multiples of t. Compare with the series in Ex. 1–2.

5. For e^θ, the cosine series in multiples of θ.

6. In a make-and-break electric circuit, if the voltage E is constant ($=20$) from $t=0$ to $t=\pi$, what Fourier sine series in multiples of t would equal E in this interval? (Simply use the result of Ex. 1.)

7. A string of a musical instrument, at the instant when plucked, had the position of the straight line, $y=.01\,x$, from $x=0$ to $x=\pi$ (in.). By Ex. 2, what cosine series would represent y in this interval?

8. Along one edge of a metal plate at a certain time, the temperature ($T°$) varied thus with the distance (x ft.): $T=100+10\,x$. By Ex. 1–2, what sine series would equal T from $x=0$ to $x=\pi$?

9. The equation of a vibrating string of a musical instrument t sec. after being plucked was:

$$y=.2[\sin x\cos kt-\tfrac{1}{9}\sin 3\,x\cos 3\,kt+\tfrac{1}{25}\sin 5\,x\cos 5\,kt-\cdots],$$

where $k=512\,\pi$. What Fourier series does this give for y when $t=0$? When $t=\pi/(4\,k)$?

10. Derive formula (87) for B_n in the sine series.

§ 165. Larger Intervals. As is shown in more advanced courses, a combination series can be obtained, viz.:

$$\begin{aligned} f(x)=A_0+A_1\cos x+A_2\cos 2\,x+A_3\cos 3\,x+\cdots \\ +B_1\sin x+B_2\sin 2\,x+B_3\sin 3\,x+\cdots, \end{aligned} \tag{89}$$

which will be valid in an interval of any length, provided the fundamental conditions are satisfied throughout. (§163.)

To secure validity from 0 to $2\,\pi$, simply use these limits in the various integrations corresponding to (79)–(84). The new formulas for the A's and B's then come out thus:

$$\begin{aligned} A_0&=\frac{1}{2\,\pi}\int_0^{2\pi}f(x)\,dx, \\ A_n&=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx, \\ B_n&=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx. \end{aligned} \tag{90}$$

The interval of integration is twice as great; the numerical factors are half as large.

To secure validity from $-\pi$ to π, use formulas like (90), with the limits of integration $-\pi$ and π.

For any other interval, a substitution is first made, which amounts to a change of scale, so as to reduce the given interval to one which runs from 0 to 2π, say.

§ **166. Broken Intervals.** A function $y=f(x)$ may be so defined that its graph consists of a set of connected or separate straight lines or curves. (Cf. Figs. 100, 101.) Different mathematical formulas or equations may be employed to express the value of y in different parts of the interval.

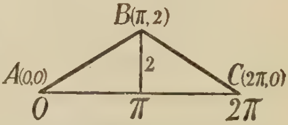


FIG. 101.

In such a case the coefficients, A's and B's, of the Fourier series are found from the same integral formulas as formerly. But each integral, say from 0 to 2π, must be split into separate integrals for the several parts of the interval, and in each separate integral a formula must be used for $f(x)$ which will correctly express y in that partial interval.

§ **167. Remarks on Chapter VI.** The uses of series fall into two main categories:

(1) To facilitate calculations involving well-known functions; *e.g.*, to compute additional values of those functions, or effect difficult integrations, etc.

(2) To define, and afford a means of using, functions not expressible by simple combinations of the elementary functions.

Most cases of the second kind lie outside the scope of this text; but one illustration is the definition of e^{x+iy}, and the deduction of its value $e^x(\cos y+i\sin y)$. In this connection, because of certain analogies, we found it convenient to define the hyperbolic functions.

The most important things to remember about these latter functions are: the exponential definitions of sinh u and cosh u, their derivatives,

the basic relation $\cosh^2 u - \sinh^2 u = 1$, the relations defining the other functions, and the general form of integration formulas (70) and (71).

Taylor series, including Maclaurin series, run in algebraic powers; Fourier series, in sines and cosines of multiple angles. A function $f(x)$ has only one Taylor series in powers of a given binomial $(x-a)$. But it may have different Fourier series in one interval.

Taylor series are used so frequently that the formula for their coefficients, both in the general case and in the special Maclaurin form, should be fixed indelibly in mind.

Note this peculiarity of the Maclaurin series for $\sin x$. This function never exceeds 1, and repeats its values forever as $x \to \infty$. Yet it is always equal to a series involving powers like x, x^3, x^5, etc., — each of which becomes infinite with x and none of which ever repeats. We should, however, bear in mind here, — as always in connection with infinite series, — that *it is not the sum of n terms which equals $f(x)$, but rather the limit approached by that sum, as $n \to \infty$.*

In case a function is given only by a table of values, or by a graph for which no simple formula is known, we can merely make approximate calculations. The method of finite differences, with its successive Δ's, replaces Taylor's series. And the coefficients of a Fourier series, given by the integrals in (85)–(87), or (90), can be found only approximately.

The most convenient methods of approximating integrals will be studied systematically in the next chapter.

EXERCISES

1. Find the Fourier series for $f(x) = 1$, valid from $x=0$ to $x=2\pi$. How many terms are present?

2. Find the Fourier series for $f(x) = x$, valid from $x=0$ to $x=2\pi$. Compare with the answers to Ex. 2, p. 277.

3. Like Ex. 2, if valid from $x=-\pi$ to $x=\pi$.

4. In Fig. 101 show that, from $x=\pi$ to $x=2\pi$, $y=4-(2/\pi)x$. What is the formula from 0 to π?

5. Find the Fourier series valid from 0 to 2π for a function defined thus: from $x=0$ to $x=\pi$, $f(x)=4$; from $x=\pi$ to $x=2\pi$, $f(x)=0$. Draw the graph of $f(x)$.

6. Like Ex. 5 for this function: 0 to π, $f(x)=2$; π to 2π, $f(x)=4$.

The following exercises are for review.

7. Find the Maclaurin series for $\cosh x$ as far as x^3. Compute cosh .002, and estimate the accuracy.

8. Find the Maclaurin series for $\sinh x$ as far as x^3. Compute

$$\int_0^{.3} \frac{\sinh x}{x}\, dx.$$

9. Find simple parametric equations in terms of hyperbolic functions for each following curve:

(a) $\dfrac{x^2}{36}-\dfrac{y^2}{64}=1$, (b) $\dfrac{y^2}{64}-\dfrac{x^2}{36}=1$.

10. On a Mercator map of the earth, any line parallel to the equator and y in. away represents a latitude circle, in latitude $L^{(r)}$, where $L=\text{gd}(y/k)$, and k is the number of inches used to represent 1 radian of longitude. If $k=1$, find L and its degree equivalent for lines $y=.5$, 1, 1.5, $\cdots$, 3.5. On a vertical line mark off the latitudes represented at points one half inch apart.

11. Look up gd (1.23).

12. (a) From Table I find log 1.387562.

(b) Find N to ten figures, if

$$\log N = .142\,276\,3002.$$

TABLE I

N	$\log N$
1.3875	.142 232 9918
876	264 2912
877	295 5883
878	326 8832

13. Table II gives the pressure of saturated steam (p kg./sq. cm.) for various temperatures ($T°$). Find p when $T=107$.

14. The interest yield r of a certain bond, — known to be somewhat over 6%, — can be calculated from the equation

$$F(x)=900\,x^{21}-925\,x^{20}-1000\,x+1025=0, \tag{91}$$

where $x=1+\frac{1}{2}r$. (a) Show that the Taylor series for $F(x)$ in powers of $(x-1.03)$, called h for brevity, is:

$$F(x)=-1.37+695.5\,h+32215\,h^2+\cdots=0. \tag{92}$$

(b) Find an approximate value for h by ignoring h^2 and higher powers. (c) Use this value of h in the h^2 term, combine with the leading term -1.37, and find h more accurately. Finally, get r.

TABLE II

T	p
80	.4828
85	.5894
90	.7149
95	.8620
100	1.0333
105	1.2319
110	1.4608

CHAPTER VII

MEAN VALUES AND APPROXIMATE INTEGRATION

PART I. MEAN VALUES

§ 168. Average Value of a Varying Quantity. In scientific work we often use the idea of an average value of a continually changing quantity. But just what does this mean? *E.g.*, if Q varies with x, what shall we regard as its true average value $\bar{Q}$ for all values of x, from $x=a$ to $x=b$? Manifestly "all" values of x cannot be listed one-by-one, and their corresponding Q's "averaged" in any literal sense.

Likewise, if Q varies with two or more independent variables, x, y, $\cdots$, a similar question arises as to the average value $\bar{Q}$ for all values of x, y, $\cdots$, in some area, or region of space, etc.

The definitions in question will be stated in detail presently. (§§ 169, 171.) In a general way we may say: If the arithmetical average of n values of Q, equally spaced throughout an interval or region, approaches a limit as $n \to \infty$, that limit will be regarded as the true average or "mean" value of Q, for all values of x, or x and y, etc., in the interval or region.

From the fuller definitions we shall deduce the following expressions for the mean value $\bar{Q}$, according as one, two, or three independent variables are involved, — either in the formula for Q or in the interval or region considered:

$$\frac{\int Q\,dx}{\int dx}, \qquad \frac{\int\int Q\,dx\,dy}{\int\int dx\,dy}, \qquad \frac{\int\int\int Q\,dx\,dy\,dz}{\int\int\int dx\,dy\,dz}. \qquad (1)$$

The integrations extend in each case over the interval or region in question.

And these expressions will be found to be consistent with the idea of mean value as commonly used in physical problems.

E.g., the average or mean force used in moving an object from $x=a$ to $x=b$ is, according to (1):

$$\bar{F}=\frac{\int_a^b F\,dx}{\int_a^b dx}=\frac{W}{b-a}, \tag{2}$$

in other words, the total work divided by the distance, — as defined in Physics.

Again, by (1), the mean density $\bar{D}$ of a solid of any varying density D is

$$\bar{D}=\frac{\iiint D\,dx\,dy\,dz}{\iiint dx\,dy\,dz}=\frac{M}{V}. \tag{3}$$

I.e., $\bar{D}$ equals the total mass, divided by the volume, — as often defined. Compare also the corresponding expressions in the case of linear and surface density. (Ex. 11, 12, p. 286.)

Moreover, (1) is consistent with the elementary definition of a mean ordinate of a graph. (Cf. *Intro.*, §§ 13, 301.)

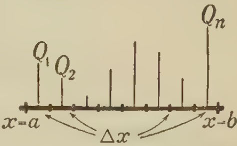


FIG. 102.

§ 169. One Independent Variable. Let Q be a function of x, continuous in an interval $x=a$ to $x=b$. Divide this interval into n equal sub-intervals Δx; and let $Q_1, Q_2, \cdots, Q_n$ be the values of Q at the middle of successive sub-intervals. The arithmetical average of these n equally spaced values of Q is

$$q_n=\frac{Q_1+Q_2\cdots+Q_n}{n}. \tag{4}$$

And the *limit* approached by q_n as $n \to \infty$, we shall call "the mean value of Q, or of $f(x)$, from $x=a$ to $x=b$":

$$\bar{Q} = \mathop{L}_{n \to \infty} q_n. \tag{5}$$

It is only in this limit sense that $\bar{Q}$ can correctly be called the average of "all" values of Q from $x=a$ to $x=b$.

That q_n actually does approach a limit, -- and what value that limit has, — can be seen as follows. Multiply the numerator and denominator in (4) by Δx, and note that $n\Delta x = b - a$.

$$\therefore \qquad q_n = \frac{Q_1\Delta x + Q_2\Delta x \cdots + Q_n\Delta x}{b-a}. \tag{6}$$

If $n \to \infty$, then $\Delta x \to 0$. And, by § 58, the numerator in (6) approaches a limit, viz. $\int_a^b Q\,dx$.

Observe, incidentally, that the values Q_1, Q_2, $\cdots$, Q_n, could have been taken *anywhere* within the sub-intervals Δx: the limit of q_n would have been the same.

Hence, for any continuous function $Q=f(x)$, the mean value from $x=a$ to $x=b$ is

$$\bar{Q} = \frac{\int_a^b Q\,dx}{b-a} = \frac{\int_a^b f(x)\,dx}{b-a}. \tag{7}$$

This establishes the first of the forms in (1). To calculate a mean value of this type we shall simply use (7) as a formula.

Ex. I. The mean value of x^4 from $x=1$ to $x=3$ is

$$\frac{\int_1^3 x^4\,dx}{3-1} = \frac{\frac{1}{5}(3^5-1^5)}{2} = 24.2.$$

[Compare this result with the value of x^4 at the beginning, middle, and end of the interval, 1 to 3.]

Ex. II. An alternating electric current varied thus: $i = 10 \sin 200\, t$. Find the mean value of i^2 during one complete cycle.

A cycle starts at $t = 0$ and ends when the angle $200\, t$ is $2\,\pi$; *i.e.*, at $t = .01\,\pi$. Hence the mean value of i^2 is

$$\frac{\int_0^{.01\pi} i^2\, dt}{.01\,\pi} = \frac{\int_0^{.01\pi} 100 \sin^2 200\, t\, dt}{.01\,\pi} = \frac{50}{.01\,\pi}\left[t - \frac{\sin 400\, t}{400}\right]_0^{.01\pi} = 50.$$

Remark. The work which an electric current will do is not determined by the average current flowing, but by the average value of the square of the current. If a constant current C is to do the same amount of work as the alternating current in Ex. II, we must have $C^2 = 50$, or $C = \sqrt{50} = 7.07$, approx. This equivalent steady current C is called the "mean effective current."

EXERCISES

1. Find the mean value of each of the following functions, in the interval specified. Check roughly by noting the values of the function at the beginning and end.

(*a*) $x^2 + 1$, $x = 1$ to 5; (*b*) t^3, $t = 2$ to 4;

(*c*) $2 \cos \theta$, $\theta = 0$ to $\frac{\pi}{2}$; (*d*) $\sqrt{x}$, $x = 4$ to 9;

(*e*) $\sinh \phi$, $\phi = 0$ to 2; (*f*) 10^y, $y = 1$ to 2;

(*g*) $\frac{1}{x^2 + 1}$, $x = 0$ to 1; (*h*) $\frac{4}{u}$, $u = 2$ to 6.

2. Find the mean current i in Ex. II above, during the first half-cycle where i was positive. Compare with the mean effective current C; also with the maximum i.

3. An electric current varies thus: $i = 20 \sin 120\, \pi t$. Find the mean effective current for a complete cycle.

4. The force F between two electric charges q and q', at a distance r apart, is $F = qq'/r^2$. Find $\overline{F}$, from $r = 5$ to $r = 10$.

5. The strength of field at a point on the perpendicular bisector of a bar magnet, and r cm. away, is

$$H = \frac{2\,ml}{(r^2 + l^2)^{\frac{3}{2}}},$$

where m and l are constants. Find $\overline{H}$, from $r = \frac{4}{3}l$ to $r = 2\,l$.

6. The same as Ex. 5 for a point in line with the magnet, if

$$H = \frac{4\,mlr}{(r^2 - l^2)^2}.$$

7. The friction of a wind exerted on the earth's surface varies thus with the velocity of the wind near the ground: $F = kv^2$. If v increases gradually from 0 to V, what is the mean friction?

8. The frictional resistance to the movement of tides over the bottom of the sea causes a dissipation of energy. The rate of loss (R ergs per sq. cm. per sec.) is $R = 3600 \cos^3\left(\frac{2\,\pi}{T}t\right)$, where T is the semi-diurnal period. Find $\overline{R}$ for the first quarter-period, $t = 0$ to $\frac{1}{4}\,T$.

9. Find the average distance from the origin or pole to the points on one loop of the curve $r = 10 \sin 2\,\theta$, for all directions.

10. Find the average distance from any one point on a circle of diameter 10 in. to all other points on the circle.

11. If D be the surface density of a thin flat plate, of area A, what would be the expression for the mean surface density, according to (1)? Compare with the physical definition.

12. The same as Ex. 11 for the linear density of a thin rod, of length L.

§ 170. Alternative Independent Variables. The idea of a mean value is vague unless we know what is to be regarded as the independent variable.

For instance, a changing force F may be regarded as varying with the time t or with the distance x that the object has been moved. The mean value of F for all *instants*, up to a total time T, is

$$\overline{F}_t = \frac{\int_0^T F\,dt}{T}. \tag{8}$$

But the mean value for all *distances*, to a total distance X, is

$$\overline{F}_x = \frac{\int_0^X F\,dx}{X}. \tag{9}$$

These two values may differ widely. (Cf. Ex. 16, p. 292.)

As another illustration, consider the distance y of any point P on a quarter-circle from either straight side. This may be regarded as a function of either x or ϕ (Fig. 103):

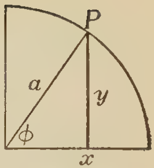


FIG. 103.

$$y = \sqrt{a^2 - x^2}, \qquad y = a \sin \phi. \tag{10}$$

The mean value $\bar{y}$ will be different in the two cases.

It might be argued superficially that the average y for all distances x should be the same as the average y for all angles ϕ, — inasmuch as taking all distances will secure all angles, and vice versa. But we must

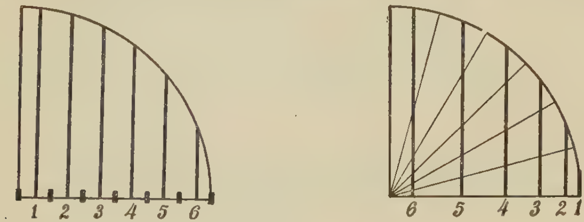


FIG. 104.

realize that we are not literally using "all" the distances. Actually, $\bar{y}$ denotes the limit of the average of n values chosen in a certain definite way. And it matters considerably *how* these n values are chosen.

E.g., Fig. 104 shows six ordinates y taken at equal intervals Δx, also six others at equal intervals $\Delta\phi$. The latter method of selection takes more ordinates from the part of the curve where y is small.

§ 171. Functions of Several Variables. Let Q be a function of two independent variables x and y, say $Q = f(x, y)$, continuous for all values of x and y which (if plotted) would fall in a given area or continuous portion R of the XY-plane.

Let R be cut into small rectangles $\Delta x \Delta y$ by lines parallel to the axes OX and OY, Δx being contained exactly in the extreme horizontal length of R, and Δy in the extreme vertical

width. (There may be some small irregular areas left over, as in Fig. 105.) The limit of the arithmetical average of the values of Q at the centers of all the rectangles whose centers lie in R, as Δx and Δy both approach zero, is called the mean value of Q in the region R, — written $\bar{Q}_R$, or simply $\bar{Q}$ if there is no ambiguity.

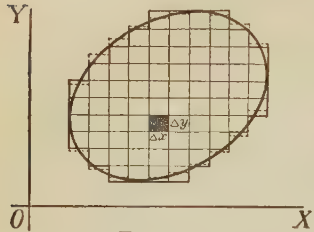


FIG. 105.

Let N be the number of suitable rectangles $\Delta x \Delta y$ present, and A_N the sum of their areas. Let n be the number of vertical rows; and m_1 the number of rectangles in the first row, m_2, in the second, etc. Let the values of Q, taken at midpoints of the rectangles, be numbered as follows: in the first row, $Q_1^{(1)}, Q_2^{(1)}, Q_3^{(1)}, \cdots, Q_{m_1}^{(1)}$; in the second row, $Q_1^{(2)}, Q_2^{(2)}, \cdots, Q_{m_2}^{(2)}$; etc. Then the arithmetical average of all N of these Q's is

$$q_N = \frac{[Q_1^{(1)} \cdots + Q_{m_1}^{(1)}] + [Q_1^{(2)} \cdots + Q_{m_2}^{(2)}] \cdots + [Q_1^{(n)} \cdots + Q_{m_n}^{(n)}]}{N}. \tag{11}$$

And the limit of q_N, as both Δx and Δy approach zero [or, for definiteness, let us say: first $\Delta y \to 0$, then $\Delta x \to 0$], is $\bar{Q}_R$.

To see what limit is approached, first multiply the numerator and denominator in (11) by $\Delta x \Delta y$. The new denominator $N\ \Delta x \Delta y$ is A_N; and as $\Delta y \to 0$, this approaches the sum A'_n of the areas of rectangles in the n strips of width Δx whose ends (dotted in Fig. 105) are bisected by the curve. As $\Delta x \to 0$, A'_n in turn approaches A, the area of R.

The new numerator for (11), after multiplication by $\Delta x \Delta y$, takes the form S:

$$S = [Q_1^{(1)} \Delta x \Delta y \cdots + Q_{m_1}^{(1)}\ \Delta x \Delta y] + [Q_1^{(2)} \Delta x \Delta y \cdots + Q_{m_2}^{(2)} \Delta x \Delta y] + \cdots. \tag{12}$$

And by the Theorem, p. 487, of the Appendix, the limit of S as $\Delta y \to 0$ and then $\Delta x \to 0$ is precisely the double integral

$$\int\int Q\, dy\, dx,$$

the integration extending over the whole region R.

Thus our definition gives finally:

$$\bar{Q}_R = \frac{\int\int Q\, dy\, dx}{A} = \frac{\int\int Q\, dy\, dx}{\int\int dy\, dx}. \tag{13}$$

As the order of integration could be reversed, this agrees with (1). Either can be used as a formula.

Evidently a wholly analogous discussion would establish the corresponding formula, in the case of a quantity Q which is a function of three variables, $Q = f(x, y, z)$. But we should then number and group the Q's by layers, columns, and blocks; and multiply numerator and denominator of the equation analogous to (11) by $\Delta x\, \Delta y\, \Delta z$. And so on.

The reasonableness of the expressions in (1) from a physical standpoint has already been pointed out. (See also Ex. 12, p. 292.)

§ **172. Other Coördinate Systems.** The mean value of Q in the region R of Fig. 105 can be expressed in another form, based upon polar coördinates, which is sometimes more convenient than (13), viz.

$$\bar{Q}_R = \frac{\int\int Q\, r\, dr\, d\theta}{\int\int r\, dr\, d\theta}. \tag{14}$$

This can be proved by a rigorous argument similar to the foregoing but considerably more involved. We content ourselves here with a rough argument, by the "free and easy method."

Regard $\tilde{Q}_R$ as the arithmetical average (q_N) of exceedingly many (N) values of Q, taken at points exceedingly near together and everywhere equally close. Then the number of points, or values of Q, taken within any area, $dA = r\, dr\, d\theta$, is proportional to that area, — say $k\, r\, dr\, d\theta$. Within any elementary area, Q is constant. Hence the sum of the values of Q within dA is $(k\, r\, dr\, d\theta)Q$. The sum of all the values of Q

within R is the sum (integral) of all these elementary sums. And the total number of values of Q taken in R is the sum (integral) of all the elementary numbers, $k\,r\,dr\,d\theta$.

$$\therefore \qquad \bar{Q}_R = q_N = \frac{\text{sum of all } Q\text{'s}}{N} = \frac{\int\int Q\,k\,r\,dr\,d\theta}{\int\int k\,r\,dr\,d\theta}. \tag{15}$$

This reduces to (14) on canceling k.

In like manner, for the case of three independent variables, instead of $dx\,dy\,dz$ in the formula

$$\bar{Q} = \frac{\int\int\int Q\,dx\,dy\,dz}{\int\int\int dx\,dy\,dz},$$

we may use dV in either spherical or cylindrical coördinates.

For, an argument like that just above expresses the number of values of Q taken at points in any element dV as $k\,dV$. Then the total number of Q's in R is $N = \int\int\int k\,dV$, and we have

$$\bar{Q}_R = q_N = \frac{\text{sum of all } Q\text{'s}}{N} = \frac{\int\int\int Q\,k\,dV}{\int\int\int k\,dV}. \tag{16}$$

All the preceding mean value formulas may be summarized *symbolically* by writing

$$\bar{Q}_R = \int Q\,dR \;\div\; \int dR, \tag{17}$$

where dR denotes the numerical measure of an element of the region R, whether a length ds, an area dA, or a volume dV. This may be expressed in any convenient coördinate system. There may be one, two, or three integrations.

Ex. I. In a metal sphere of diameter 20 cm. the temperature $T°$ at any distance D cm. from the lowest point was at certain instant: $T = 300 - 12\,D$. Find $\bar{T}$ for all points in the sphere at that time.

The region being three-dimensional, we must take dR as dV. In spherical coördinates, $D=\rho$, whence $T=300-12\,\rho$.

$$\therefore\ \bar{T}=\frac{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{20\cos\phi}(300-12\,\rho)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}{V[\text{or } \frac{4}{3}\pi(10^3)]}.$$

This comes out finally: $\bar{T}=156$. [Observe that, at the hottest point, $D=0$ and $T=300$. At the coolest point, $D=20$ and $T=60$.]

EXERCISES

1. Find the mean value of the function x^2+y^2 in each specified portion of the XY-plane:

(*a*) The rectangle whose sides are $x=0$, $y=0$, $x=4$, $y=2$;

(*b*) The triangle whose sides are $x=0$, $y=0$, $x+2\,y=4$;

(*c*) The region within the circle $x^2+y^2=100$.

(*d*) Find the mean value in (*c*) by using polar coördinates.

2. Find the mean value of each of the following functions in the specified portions of the XY-plane:

(*a*) $x+y$, within the first quadrant of the circle $x^2+y^2=100$;

(*b*) $4\,y^3$, within the first quadrant of the circle $x^2+y^2=4$;

(*c*) $r\cos^3\theta$, within the circle $r=6\cos\theta$;

(*d*) r^2, within the circle $r=2\cos\theta$;

(*e*) $\sin\theta$, within the circle $r=8\sin\theta$.

3. In a rectangular plate, 8 in. by 4 in., the temperature at any point (x, y) is $T=150+10\,x+6\,y-x^2-y^2$, where x is measured from a short side and y from a long side. Find $\bar{T}$ for the entire plate.

4. The same as Ex. 3 for a circular plate of radius 10 in., if $T=250-r^2$, where r is measured from the center.

5. In a water pipe of radius 7 in., the velocity of flow (V in./sec.) at any distance (r in.) from the center is $V=50-r^2$. Find $\bar{V}$ for all points of a cross section; also for all points of a diameter of the section. Why should the values differ?

6. Find the mean distance from a vertex of a square, of side 5 in., to all interior points.

7. Find the mean value of the function $x+y+z$ at all points in a rectangular space bounded by the coördinate planes and the planes $x=3$, $y=4$, $z=5$.

8. Find the mean value of z within a cone whose vertex is at the origin, and whose circular base is 10 in. higher and of radius 5 in.

9. Find the mean value of ρ in a sphere of radius 10 in., ρ being measured from the center.

10. The density of a sphere of radius 4, at a distance ρ from the center, is $D=12-\frac{1}{2}\rho$. Find the mean density $\overline{D}$: (a) for the entire sphere; (b) for all points in a circular section through the center; (c) for all points of a radius. Why should these values be different?

11. Inside a hollow sphere the illumination varies thus with the spherical coördinates, the origin being at the lowest point and the Φ-axis upward:

$$I=\frac{1+4\cos\phi}{\rho^2}.$$

Find $\bar{I}$ for all interior points.

12. By (1) or (13) express the average height of a surface $z=f(x, y)$ above all points of the XY-plane which lie below it. Interpret the expression geometrically.

13. Answer this question without calculation: The mean height $\bar{z}$ of a hemispherical surface above its flat base is to be found: (a) for all points of the base, and (b) for all points of the surface. Which must be the greater, and why?

14. Find the mean value of y in Fig. 103, using x as the independent variable. Likewise using ϕ. [See equations (10).]

15. If $y=x^2$ and $x=t^3$, find the mean value of y for all values of t, from $t=0$ to $t=1$. Also for all values of x, from $x=0$ to $x=1$.

16. In a simple harmonic motion the distance x varied thus: $x=10\sin 2t$. Express the acceleration A as a function of t; also as a function of x. Find its mean value $\bar{A}$ in each case, from the start until x first reached a maximum.

§ **173. Centroids and Mean Values.** The z-coördinate of the centroid of a solid of constant density, $D=k$, is given by

$$z=\frac{\iiint z\,dm}{\iiint dm}=\frac{\iiint z\,k\,dz\,dy\,dx}{\iiint k\,dz\,dy\,dx}. \tag{18}$$

On cancelling k this reduces to the expression for the mean value of z for all points of the solid. Thus the ordinate of the centroid is the average of the ordinates of all the particles;

and it is immaterial which is regarded as the meaning of $\bar{z}$. The same is true of $\bar{x}$ and $\bar{y}$.

For a non-homogeneous solid the same idea will hold if we regard the varying density as expressing the number of ultimate particles of matter in an element of volume dV.

Center of Population. It is interesting to note that the census reports of the United States define the center of population, roughly, by using the idea of a (not perfectly analogous) center of gravity; and define it accurately by using the idea of mean values. Thus, the center is that point whose latitude $\bar{y}$ is the mean of the latitudes for all the inhabitants; and whose distance ($\bar{x}$ mi.) from a chosen meridian is the mean of such distances for all the inhabitants.

If P be the total population, and ΔP that in any small area ΔA, regarded as concentrated and having a single x and a single y, and if n be the total number of such small areas, then

$$\bar{x}=\frac{x_1\Delta P_1+x_2\Delta P_2\cdots+x_n\Delta P_n,}{P} \tag{19}$$

and similarly for $\bar{y}$. Double integrals could be substituted for the sum in (19), if it were physically possible to subdivide the population indefinitely and let $\Delta A \to 0$.

§ 174. Astronomical Illustrations. Mathematical Astronomy furnishes various illustrations of mean values. An interesting case is the mean distance of a planet from the sun. Ignoring minor deviations, the radius vector r varies thus with the polar angle θ:

$$r=\frac{a(1-e^2)}{1+e\cos\theta}, \tag{20}$$

where a is the major semi-axis of the elliptic orbit, and e is the "eccentricity" $[=\sqrt{1-(b^2/a^2)}]$.

(I) The mean value of r for all *directions* in a half-revolution is

$$\bar{r}_\theta=\frac{\int_0^\pi r\,d\theta}{\int_0^\pi d\theta}=\frac{a(1-e^2)}{\pi}\int_0^\pi\frac{d\theta}{1+e\cos\theta}. \tag{21}$$

By formula (72), p. 497, this reduces to $a\sqrt{1-e^2}$, which equals b. Thus the mean r, from this standpoint, equals the minor semi-axis.

(II) The mean value of r for all *instants* in a half-period, $\frac{1}{2}T$, is

$$\bar{r}_t = \frac{\int_0^{\frac{1}{2}T} r\,dt}{\int_0^{\frac{1}{2}T} dt}. \tag{22}$$

There is no simple formula for r as a function of t. But the integrals in (22) can be calculated by using an astronomical relation established later (§ 289), viz.

$$dt = \frac{r^2 d\theta}{h}, \tag{23}$$

where h is a certain constant. Thus (22) becomes, on canceling h and changing limits appropriately:

$$\bar{r}_t = \frac{\int_0^{\pi} r^3\,d\theta}{\int_0^{\pi} r^2\,d\theta}. \tag{24}$$

By formulas (73), (74), p. 497, this reduces to

$$\bar{r}_t = [\pi a^3\sqrt{1-e^2}(1+\tfrac{1}{2}e^2)] \div [\pi a^2\sqrt{1-e^2}], \text{ or } \bar{r}_t = a(1+\tfrac{1}{2}e^2).$$

Neither $\bar{r}_\theta$ nor $\bar{r}_t$ equals a, which is called the planet's "mean distance" from the sun. But a is the mean value of r on the basis of all *distances* reached in the orbit.

Observe that $\bar{r}_t$ exceeds $\bar{r}_\theta$. Astronomically, this should be expected; for a planet has the greatest angular speed when closest to the sun. Consequently, n values of r chosen at equal intervals of time Δt would include fewer of the small values than would n values chosen at equal intervals $\Delta\theta$.

EXERCISES

1. Show from (23) that the angular speed is greatest when r is least, and vice versa.

2. Show from (20) that the greatest and least values of r are $a(1+e)$ and $a(1-e)$. What is their average? Show that this is also the mean value of r "for all values of r in the orbit." (§ 174.)

3. The force with which the sun attracts a planet may be expressed as $F=k/r^2$. Find the mean force $\overline{F}$:

(*a*) For all values of r in the orbit;

(*b*) For all values of θ in a half-revolution;

(*c*) For all values of t in a half-period.

4. Find the average distance from a point on a circle of radius 10 to all interior points; also to all points on the circle.

5. A county is rectangular, 20 mi. by 10 mi. Find the average distance from its center to all of its points.

6. Find the mean value of the quantity $\sin\phi$ for all points on the surface of a hemisphere, ϕ being measured from the vertical radius. [See Ex. 3, p. 245, as to dS.]

7. Find the mean distance from the South Pole of the earth to all interior points, regarding the earth as a sphere with radius $a=3960$.

8. Find the mean distance from the South Pole (through the earth) to all points on the surface. [Take the origin at the center, and show that $D=2a\cos(\phi/2)$. Cf. Ex. 6.]

9. The average federal income tax, for all different amounts of taxable personal income up to \$10,000, probably differs considerably from the average tax paid by all the individuals concerned. Why?

10. A man's "insurance age" is his age at his nearest birthday. This determines the premium he must pay in taking out a life insurance policy, say for \$1000. The average premium charged by a company for such a policy, for all insurance ages from 30 to 60, differs from the average premium paid by all men who take out such a policy within those ages. What accounts for this?

PART II. APPROXIMATE INTEGRATION

§ 175. The Problem. Many integrals encountered in scientific work cannot be found exactly. An instance is

$$\int\sqrt{1-x^4}\,dx.$$

There is no finite combination of elementary functions, such as radicals, logarithms, trigonometric functions, etc., whose derivative is $\sqrt{1-x^4}$.

Any definite integral, $\int_a^b f(x)\,dx$ is defined (§ 59) as the limit of a certain sum, of the form

$$f(x_1)\Delta x+f(x_2)\Delta x \cdots +f(x_n)\Delta x. \tag{25}$$

The integral exists if that sum approaches a limit as $\Delta x \to 0$ and $n \to \infty$. In this sense the function $f(x)$ is "integrable"; and the integral may be regarded as some function of a or b, or both. But $f(x)$ may be non-integrable in the sense that no finite combination of the standard elementary functions will exactly express its value. In such a case, we resort to approximation.

One method is to regard the integral as equal to the sum (25) above, rather than the limit of that sum. Using several intervals of some short length Δx, we choose a convenient value x_1, x_2, etc., in each, calculate the values of the function for those values of x, and form the sum (25).

Other methods, usually preferable, are: (A) By graphical areas; (B) By Simpson's Rule; (C) By expansion into series; (D) By using special tables, prepared in various ways. Some of these methods have been touched upon previously, but they will be summarized here for completeness and review.

§ **176. Graphical Areas.** To approximate any definite integral

$$\int_a^b f(x)\,dx, \tag{26}$$

simply plot the graph of the function

$$y=f(x),$$

and measure the area under the curve from $x=a$ to $x=b$.

This can be done fairly accurately by calculating many points, using a large scale on well ruled paper, and reading mean ordinates as in *Intro.*, § 13.

The area can also be measured by an instrument called a planimeter. By running the tracing point of the instrument around the boundary several times, and averaging results, accuracy sufficient for most engineering work may be secured.

§ **177. Simpson's Rule.** As explained in the *Introduction*, pp. 409–411, the mean ordinate in any interval, $x=a$ to $x=b$, can be found for many curves by this rule: Average the heights at a and b with four times the height at the middle.

I.e.,
$$\bar{y}=\tfrac{1}{6}(y_a+y_b+4\,y_m). \tag{27}$$

The area under the curve is this $\bar{y}$ times the base $(b-a)$.

Thus for any function $f(x)$ which is to be integrated, we find its values at the beginning, end, and middle of the interval, and use the formula

$$\int_a^b f(x)dx=\tfrac{1}{6}[f(a)+f(b)+4\,f(x_m)](b-a), \tag{28}$$

where x_m is the value of x midway between a and b.

Formula (28) is exact if $f(x)$ is a rational algebraic function of degree 1, 2, or 3. (*Intro.*, p. 490.) It is very approximate in many other cases, especially if the interval be small.

A large interval, a to b, may be split into several small intervals, the rule applied to each, and the results added.

Ex. I. Approximate the integral: $Q=\int_1^5 \frac{1}{x}\,dx.$ (In this illustration we can check by actual integration.)

(*A*) *Using a single interval.* The values of $1/x$ at $x=1$ and 5, and four times the value at $x = 3$, are, respectively:

$f(1)=1,\qquad f(5)=\tfrac{1}{5}=.2,\qquad 4\,f(3)=\tfrac{4}{3}=1.33333.$

Hence $Q=\tfrac{1}{6}[1+.2+1.33333](5-1)=1.68889$, approx.

(*B*) *Using four intervals.* For the first interval, 1 to 2:

$$f(1)=1, \qquad f(2)=.5, \qquad 4\,f(1.5)=\tfrac{8}{3}=2.66667.$$

This gives as the integral for the partial interval:

$$\tfrac{1}{6}[1+.5+2.66667](2-1)=.69444.$$

For the second, third, and fourth intervals the corresponding results are .40556, .28770, and .22315. The total for all four intervals is $Q=1.61085$. This differs considerably from the result in (*A*), — viz. by .07804.

(*C*) *Using ten intervals.* The result is found to be $Q=1.60949$. This differs from the result in (*B*) by .00136, — appreciable but much less than the change from (*A*) to (*B*).

(*D*) *Using twenty intervals.* The result is $Q=1.60944$. The change this time is only .00005, and we assume that we are now pretty close to the true value. Indeed, integration gives $Q=\log 5-\log 1=1.60944-$.

A Short Cut. When using many intervals, it is helpful to collect all the middle values which have to be multiplied by 4; also all the end values (aside from the very first and last), as each occurs in two successive intervals and is therefore to be doubled. By using this idea, and by reading the values of $1/x$ from a table of reciprocals, the results in (*C*) and (*D*) above were calculated in a few minutes. For case (*C*) the work was arranged as shown to the right.

Both sets of values used here appear again in case (*D*), — as end values, — and their sums can be used there immediately with suitable multipliers.

End Values (Except $x=1,\ 5$)		*Mid-Values*	
x	$\frac{1}{x}$	x	$\frac{1}{x}$
1.4	.714 286	1.2	.833 333
1.8	.555 556	1.6	.625 000
2.2	.454 545	2.0	.500 000
⋮	⋮	⋮	⋮
4.6	.217 391	4.8	.208 333
sum	3.455 097	sum	4.008 027
×2	6.910 194	×4	16.032 108
$x=1$	1.000 000	→	8.110 194
$x=5$	.200 000	Total	24.142 302
sum	8.110 194	÷6	4.023 717

Multiply this last value, the sum of all y's, by the interval .4:

$$4.023717 \times .4 = 1.6094868.$$

§ 178. Note on Tabulated Functions. To differentiate or integrate a function which is given only by a table of values, we generally use a graphical method. The slope of the graph at any point, measured approximately by drawing an apparent tangent line, gives the derivative. The area gives the integral.

In the *Introduction*, Chapter I, when we found instantaneous rates graphically or found momentum from a force-time graph, etc., we were in reality differentiating or integrating the tabulated functions.

Sometimes it is preferable to discover a formula for the table, and then differentiate or integrate. There are simple tests for a linear formula, C. I. L., Power Law, or Polynomial Law of degree n. (Cf. *Intro.* §§ 175, 323–24.)

Fourier Coefficients. A fairly accurate Fourier series for a tabulated function can be found by approximating the integrals which define the coefficients [*e.g.*, (85)–(87), p. 276]. To find any integral involving, say, $f(x)\sin nx\,dx$, simply multiply each tabulated value of $f(x)$ by the corresponding value of $\sin nx$, and then plot or use Simpson's Rule.

EXERCISES

1. Find by Simpson's Rule, using a single interval:

(*a*) $\int_1^7 x^2\,dx$, (*b*) $\int_0^2 (x^3-7x+10)dx$.

Check each result by integration.

2. Approximate the following integrals by Simpson's Rule, using first one interval and then two:

(*a*) $\int_1^9 \sqrt{x}\,dx$, (*b*) $\int_0^8 e^x\,dx$,

(*c*) $\int_0^{\frac{\pi}{3}} \cos x\,dx$, (*d*) $\int_0^4 \frac{1}{x^2+1}\,dx$.

Check by finding the true value in each case.

3. (*a*) – (*d*). Approximate the integrals in Ex. 2 (*a*)–(*d*) graphically.

4. Approximate each of the following integrals by Simpson's Rule, using two intervals:

(a) $\int_0^2 e^{-x^2}\,dx$, (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$,

(c) $\int_0^{.8} \frac{\cos x\,dx}{1+x^2}$, (d) $\int_1^2 \tan^{-1} x \log x\,dx$.

5. Approximate by Simpson's Rule, using 6 intervals:

(a) $\int_0^{1.2} \text{gd}\,(x)\,dx$, (b) $\int_0^{\frac{\pi}{3}} \sqrt{\sin x}\,dx$.

6. Plot the adjacent table smoothly, without unnecessary turns. Find graphically the value of dy/dx at $x=5$; also the value of $\int_0^9 y\,dx$.

x	y
0	0
1	8
2	28
3	54
4	80
5	100
6	108
7	98
8	64
9	0

7. In Ex. 6 find a formula for y in terms of x. (*Intro.*, § 324.) Check the answers to Ex. 6 accurately.

8. What meaning have the derivative and integral in Ex. 6:

(a) If y denotes the linear density of a rod, and x denotes the distance from one end?

(b) If y denotes the power being used in a factory and x denotes time?

§ **179. Integration by Series.** Another method of approximate integration already mentioned (p. 251, and *Intro.*, pp. 429–33) is to expand the given function into a series of terms by Maclaurin's or Taylor's method, or by the Binomial Theorem, and then integrate each term separately. Sometimes this method is preferable to Simpson's Rule and sometimes not: much depends upon the tables available for the given function.

If the expansion is easily obtained and the successive terms decrease rapidly in value, the series method has these advantages: (1) It gives the *indefinite integral*, in which we can readily substitute any number of values for x. (2) To improve the approximation at any stage it is easy to add further terms, whereas in using Simpson's Rule we must make a recalculation, using a larger number of intervals.

Some higher functions are *defined* by means of a series, and are naturally integrated in that form.

Much time can be saved if we permanently remember the form of the Maclaurin series for $\sin u$, $\cos u$, and e^u; also the Binomial Theorem:

$$(a+u)^n = a^n + na^{n-1}u + \frac{n(n-1)}{2!}a^{n-2}u^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}u^3 + \cdots. \qquad (29)$$

An example of the use of this theorem follows. (See also *Intro.*, § 319.)

Ex. I. Evaluate the integral, $F = \int_0^{\frac{\pi}{6}} \frac{dz}{\sqrt{1-.04\sin^2 z}}$.

The integrand, without dz, is of the form $(a+u)^n$, with

$$a=1, \quad u=-.04\sin^2 z, \quad n=-\tfrac{1}{2}.$$

By (29) its terms run thus:

$$1+(-\tfrac{1}{2})(1)(-.04\sin^2 z)+\frac{-\frac{1}{2}(-\frac{3}{2})}{2!}(1)(-.04\sin^2 z)^2+\cdots.$$

$$\therefore \frac{1}{\sqrt{1-.04\sin^2 z}} = 1+.02\sin^2 z + .0006\sin^4 z + \cdots. \qquad (30)$$

Integrating term by term, we find

$$F = \Big[z + .01(z-\tfrac{1}{2}\sin 2z) + .00015(-\sin^3 z\cos z + \tfrac{3}{2}\overline{z-\tfrac{1}{2}\sin 2z}) + \cdots\Big]_0^{\frac{\pi}{6}}.$$

Substituting the limits and reducing:

$$F = .5236 + .01(.0906) + .00015(.0276) + \cdots = .5245.$$

§ 180. Special Tables. Some types of integrals which occur frequently in scientific work have been calculated by one of the foregoing methods, or by some other limit process not discussed in this course; and the results have been recorded in tables. To find the value of such an integral, we simply consult the table, interpolating if necessary. A few important illustrations will be mentioned here.

(*A*) *Elliptic Integrals.* An extensive class of algebraic and trigonometric integrals, — known as elliptic integrals

because first encountered in finding the length of an arc of an ellipse, — can be reduced to three rather simple type forms.* Two of these, called the type elliptic integrals of the first and second kinds, are, respectively:

$$\text{(I)} \qquad F(k, \phi) = \int_0^{\phi} \frac{dz}{\sqrt{1-k^2 \sin^2 z}}, \tag{31}$$

$$\text{(II)} \qquad E(k, \phi) = \int_0^{\phi} \sqrt{1-k^2 \sin^2 z}\, dz. \tag{32}$$

Here $k<1$ numerically, and ϕ lies between zero and $\pi/2$ (inclusive). [The type integral of the third kind will not be given here.]

The small tables on page 499 of the Appendix give values of these integrals (I) and (II) for several values of k and ϕ. For instance, for $k = .2$ and $\phi = \pi/6$, we read

$$F\left(.2, \frac{\pi}{6}\right) = \int_0^{\frac{\pi}{6}} \frac{dz}{\sqrt{1-.04 \sin^2 z}} = .525. \tag{33}$$

This agrees with the value .5245 found in Ex. I, § 179.

To find an integral of the form (I) or (II) but having the lower limit ϕ_1, instead of zero, regard it as the difference of two like integrals, one running from 0 to ϕ and the other from 0 to ϕ_1. *E.g.*,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1-\tfrac{1}{4}\sin^2 z}\, dz = \int_0^{\frac{\pi}{3}} \sqrt{1-\tfrac{1}{4}\sin^2 z}\, dz - \int_0^{\frac{\pi}{6}} \sqrt{1-\tfrac{1}{4}\sin^2 z}\, dz$$

$$= E\left(.5, \frac{\pi}{3}\right) - E\left(.5, \frac{\pi}{6}\right).$$

The tables can be calculated by the series method. But if k is near 1, the convergence is very slow; and a transformation method, shown in higher courses, is used instead.

Larger tables, by the way, usually run according to values of an angle θ whose sine is k, rather than values of k itself. But the change in either direction is easy.

* Cf. H. Hancock, *Elliptic Integrals.*

(*B*) *The Probability Integral.* Another important integral, whose significance will be discussed in § 189, is this:

$$P(X)=\frac{1}{\sqrt{2\pi}}\int_0^X e^{-\frac{1}{2}x^2}\,dx. \tag{34}$$

The value of this "probability integral" is given on p. 501 for several values of the upper limit X. It can be calculated by the series method, but this is very tedious when X is large.

An integral of the same form but with a different lower limit can be regarded as the difference of two values of $P(X)$.

(*C*) *The Gamma Function.* Still another integral tabulated in the Appendix, which is reserved for later definition and use because of its infinite limit, is

$$\int_0^\infty x^p e^{-x}\,dx. \qquad \text{(Cf. § 188.)}$$

Ex. I. Find the length of an arc of the ellipse

$$x=10\cos\phi, \qquad y=6\sin\phi,$$

from $\phi=0$ to $\phi=\frac{\pi}{6}$.

Here $ds^2=dx^2+dy^2=(100\sin^2\phi+36\cos^2\phi)d\phi^2=(100-64\cos^2\phi)d\phi^2$.

$$\therefore \qquad s=10\int_0^{\frac{\pi}{6}}\sqrt{1-.64\cos^2\phi}\,d\phi. \tag{35}$$

To bring this to a type form, let $\phi=\frac{\pi}{2}-z$. Then $\cos\phi=\sin z$. When $\phi=0$, $z=\frac{\pi}{2}$; when $\phi=\frac{\pi}{6}$, $z=\frac{\pi}{3}$. But $d\phi=-dz$, which reverses the order of limits.

$$\therefore \qquad s=10\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sqrt{1-.64\sin^2 z}\,dz. \tag{36}$$

This being the difference $E\left(.8,\frac{\pi}{2}\right)-E\left(.8,\frac{\pi}{3}\right)$, we find $s=10(1.278-.940)$ $=3.38$. Reflection will show this value to be reasonable.

EXERCISES

1. Look up the values of the following:

(a) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{dz}{\sqrt{1-.09\sin^2 z}}$, (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-\frac{1}{4}\cos^2 t}}$,

(c) $\displaystyle\int_{\frac{\pi}{9}}^{\frac{\pi}{6}} \frac{d\phi}{\sqrt{1-.16\sin^2 \phi}}$, (d) $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1-.04\sin^2 z}\, dz$,

(e) $\displaystyle\int_0^{\frac{7\pi}{18}} \sqrt{1-.01\cos^2 t}\, dt$, (f) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{1-\frac{1}{4}\cos^2 \phi}\, d\phi$,

(g) $\displaystyle\frac{1}{\sqrt{2\pi}}\int_0^{.4} e^{-\frac{1}{2}x^2}\, dx$, (h) $\displaystyle\frac{1}{\sqrt{2\pi}}\int_{.6}^{1} e^{-\frac{1}{2}u^2}\, du$,

(i) $\displaystyle\int_0^{1.2} e^{-\frac{1}{2}t^2}\, dt$, (j) $\displaystyle\int_{.2}^{.8} e^{-\frac{1}{2}y^2}\, dy$.

2. Interpolate to find the following, — using second differences wherever they would affect the last decimal place:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dz}{\sqrt{1-\frac{9}{16}\sin^2 z}}$, (b) $\displaystyle\int_0^{\frac{13\pi}{45}} \frac{dt}{\sqrt{1-.04\sin^2 t}}$,

(c) $\displaystyle\frac{1}{\sqrt{2\pi}}\int_0^{.87} e^{-\frac{1}{2}x^2}\, dx$, (d) $\displaystyle\int_0^{\frac{\pi}{3}} \sqrt{1-\frac{1}{9}\sin^2 t}\, dt$,

(e) $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1-.49\sin^2 z}\, dz$, (f) $\displaystyle\frac{1}{\sqrt{2\pi}}\int_1^{1.25} e^{-\frac{1}{2}u^2}\, du$.

3. Expand $\sqrt{1-x^3}$ by the Binomial Theorem, and calculate to four decimals $\displaystyle\int_0^{\frac{1}{2}} \sqrt{1-x^3}\, dx$.

4. Calculate approximately the integrals:

(a) $\displaystyle\int_0^{.2} \frac{dx}{\sqrt{1-x^4}}$, (b) $\displaystyle\int_0^{.3} \frac{dt}{\sqrt{1+t^3}}$, (c) $\displaystyle\int_0^{\frac{\pi}{3}} \sqrt{1-.1\cos^2\theta}\, d\theta$.

5. Write by inspection the Maclaurin series for the function in each following integrand, and calculate the integral to four decimals:

(a) $\displaystyle\int_0^{.3} \sin x^2\, dx$, (b) $\displaystyle\int_0^{.5} \cos x^2\, dx$, (c) $\displaystyle\int_0^{1} e^{-x^2}\, dx$.

6. Calculate the following by series, and check by the tables:

(a) $\int_0^{\frac{4\pi}{9}} \sqrt{1-.09\sin^2 z}\,dz$, (b) $\int_0^{\frac{\pi}{3}} \frac{dz}{\sqrt{1-.5\sin^2 z}}$,

(c) $\frac{1}{\sqrt{2\pi}}\int_0^{.5} e^{-\frac{1}{2}t^2}\,dt$.

7. Find each of the following by the series method and also by Simpson's Rule, using two intervals:

(a) $\int_0^1 e^{-\frac{1}{2}x^2}\,dx$, (b) $\int_0^{.4} \frac{\log(1+x)}{1-x}\,dx$.

8. Find the integral in Ex. 7 (a) also by the graphical method, and check further by the table.

9. Find the length of an arc of the ellipse $x=5\cos\phi$, $y=4\sin\phi$, from $\phi=0$ to $\phi=\frac{\pi}{6}$; also the entire circumference.

10. The length of a quadrant arc of the lemniscate $r^2=100\cos 2\theta$ can be reduced to the form:

$$s=5\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{dz}{\sqrt{1-\frac{1}{2}\sin^2 z}}.$$

Find the entire length of this lemniscate.

11. The time (t sec.) required for a simple pendulum of any length (l ft.) to swing from its lowest point up to any angle θ is

$$t=\sqrt{\frac{l}{g}}\int_0^{\phi}\frac{dz}{\sqrt{1-\sin^2(A/2)\sin^2 z}},$$

where g is the gravitational acceleration, A is the greatest angle reached, and $\sin\phi=\sin\frac{\theta}{2}\div\sin\frac{A}{2}$. If $l=2.01$, $g=32.16$, and $A=60°$, find the time required to reach an angle of 30°; also one of 60°. How long is required for a complete swing?

12. As in Ex. 11 find the time of a complete swing if $l=.25$, $g=32.1$, and $A=20°$.

PART III. INDETERMINATE FORMS

Before proceeding with integration we must digress briefly to consider another important matter.

§ 181. Functions Undefined at a Point. Sometimes a functional expression assumes a meaningless form at some

point. *E.g.*, suppose that y is defined as a function of x by the equation

$$y=\frac{2^x-1}{x}. \tag{37}$$

When $x=0$, this takes the form $y=0/0$, which is meaningless and does not define any value for y.

It would be possible to regard all numbers as quotients for $0 \div 0$, since any number times the divisor (0) would give the dividend (0). But no number can be called *the* quotient. And, in the axioms of algebra, it is agreed not to call any number a quotient, but to exclude division by zero entirely.

Whenever an expression takes a meaningless form, and thus does not specify what value the function shall have at the point in question, we ask: What limit does the function *approach* as we approach that point? That limit, if there be any, we then take or define as the value of the function *at the point*, in place of the meaningless form.

Sometimes the limit is obvious; but often it requires investigation. *E.g.*, in (37) above as $x \to 0$, both numerator and denominator become very small; and the result of dividing one small number by another may be large or small. The shrinking numerator tends to make y small; the shrinking denominator tends to make y large. Without investigation no one can predict the result of these conflicting tendencies.

In the case above the limit happens to be log 2, or .6931. (Cf. p. 311, Ex. 12.) So we assign the value .6931 to y when $x=0$, in place of the meaningless 0/0. Since the value of y at $x=0$ (so defined) is the same as the value approached by y when $x \to 0$, the graph will have no break at $x=0$. The function, so defined at $x=0$, is said to be "continuous" there.

In general, a limit is doubtful for a fraction $y=f(x)/F(x)$, if both f and F approach zero. So is it, if both approach infinity; for then also there are conflicting tendencies defying prediction. Moreover, various expressions besides fractions

have doubtful limits, — as we shall see. A general method will be developed shortly for discovering what the actual limit is in each case. (§ 183.)

§ **182. Indeterminate Forms.** When a functional expression assumes a meaningless form, and the corresponding limit is doubtful without investigation, because of conflicting tendencies, the form is said to be "indeterminate."

E.g., the form $\frac{0}{0}$ assumed by $y=\frac{2^x-1}{x}$ when $x=0$ is indeterminate. Likewise the form ∞/∞ assumed by $y=\tan 3x/\tan x$ when $x=\pi/2$ is indeterminate.

Any expression in which ∞ appears (such as $\infty+10$, or $20\div\infty$, or $e^{-\infty}$, etc.) is meaningless, taken literally; and does not equal any number. "Infinity" is not a number or value, and cannot be combined with numbers algebraically.

But an expression involving ∞ is not necessarily indeterminate. For the corresponding limit of the function may be obvious.

The operation of finding the limit of a function corresponding to a meaningless form assumed at some point, is called "evaluating the form" in question. This does not mean, however, that we somehow discover a hidden value belonging to a meaningless form (which has no value). Rather, we discover what value we shall wish to *assign* to the form, in defining the function at the point.

Ex. I. Discuss $y=\frac{x}{\log x}$ at $x=0$.

There is no logarithm of zero, for e cannot be raised to any power which will give zero. But e to a very high negative power will give a very small number, — the higher, the smaller. So we write *symbolically* $\log 0=-\infty$ and

$$y=\frac{0}{-\infty}.$$

This is meaningless, of itself. But as $x\to 0$, the numerator grows indefinitely small and the denominator indefinitely

large, both of which tend to make $y \to 0$. The limit is clear; and $0/-\infty$ is not indeterminate. We define the value of y at $x=0$ as zero.

Ex. II. Discuss $y=(\sin x)^x$ at $x=0$.

This takes the form $y=0^0$, which we consider meaningless, as remarked below. Further, the corresponding limit of y is doubtful. For as the exponent (x) approaches zero, it tends to make $y \to 1$. But as $\sin x \to 0$, this tends to make $y \to 0$ or $y \to \infty$, according as the exponent is $+$ or $-$. Thus the form 0^0 is indeterminate and requires study.

Remark. The reasons for defining x^0 as 1 (*Intro.*, p. 99) do not apply to the case 0^0; and we shall regard this form as an exception to the otherwise general rule about a zero exponent. In many cases, however, "evaluating" 0^0 gives 1 as the result.

Ex. III. Discuss the form 0^∞.

This form is assumed by $y=f^F$ when $f=0$ and $F=+\infty$. Arising thus, it is meaningless but not indeterminate. For when $f<1$, the increase in F tends to make y small, — as does also the decrease in f. The limit is clearly zero.

The form 0^∞ might arise in other ways; for instance, from $y = 0^F$, the fixed number zero being raised to an increasing power. Then y would be constantly zero. The manner in which such a form as 0^∞ arises must be known in order to discuss it safely.

§ **183**. **General Method of Evaluation.** Consider first the case $0/0$, so called. Accurately stated, we seek the limit as $x \to a$ of the fraction

$$\frac{f(x)}{F(x)}, \qquad \text{given } f(a)=0,\ F(a)=0. \tag{38}$$

Assume that $f(x)$ and $F(x)$ have valid Taylor series:

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots, \tag{39}$$

$$F(x)=F(a)+F'(a)(x-a)+\frac{F''(a)}{2!}(x-a)^2+\cdots.$$

By (38), the first terms drop out. Both $f(x)$ and $F(x)$ then have a factor $(x-a)$ throughout. Canceling this common factor in the numerator and denominator of the fraction $f(x) \div F(x)$ gives:

$$\frac{f(x)}{F(x)} = \frac{f'(a)+\frac{f''(a)}{2!}(x-a)+\cdots}{F'(a)+\frac{F''(a)}{2!}(x-a)+\cdots}. \tag{40}$$

Now let $x \to a$, and the limiting value is seen to be $f'(a) \div F'(a)$, unless the latter derivative is zero. In other words, the limit of the original fraction $f(x) \div F(x)$ can usually be found by *differentiating numerator and denominator separately, and substituting $x=a$ in the resulting fraction.*

If $f'(a)=0$ while $F'(a) \neq 0$, the limit is zero; in the reverse case, infinity. If $f'(a)$ and $F'(a)$ are both zero, those terms also are absent from (40). We then cancel another $(x-a)$, and get as our limit $f''(a) \div F''(a)$, with similar exceptions.

Thus if the fraction $f'(a)/F'(a)$ which would result from the first differentiation is indeterminate, we again differentiate, separately. And so we continue, as long as *both* numerator and denominator are zero, *i.e.*, as long as we have the form $0/0$. Evidently we must not continue farther than that, as no more factors $(x-a)$ could be removed.

This method applies even when a general Taylor series is not valid, provided $f(x)$ and $F(x)$, and their derivatives as far as used, are continuous *at and near* $x=a$. It also applies to cases in which $f(x)/F(x)$ takes the form $0/0$ as x becomes infinite. (Cf. Appendix, p. 486.)

Ex. I. Find $\underset{x \to 0}{\mathrm{L}} \dfrac{\sin 3x}{e^x - 1}$.

At $x=0$, both $\sin 3x$ and e^x-1 are zero. Differentiating numerator and denominator separately, once:

$$\underset{x \to 0}{\mathrm{L}} \frac{\sin 3x}{e^x-1} = \underset{x \to 0}{\mathrm{L}} \frac{3\cos 3x}{e^x} = 3.$$

EXERCISES

1. If $y=f(x)/F(x)$, and both f and F approach infinity, why is the limit of y doubtful? (Indicate whether f alone tends to make y large or small. Likewise F alone.)

2. The same as Ex. 1 for $y=f^F$ if $f\to 1$ and $F\to\infty$.

3. For which of the following expressions is the limit doubtful? Tell why. (Give the limit, finite or infinite, if not doubtful.)

(*a*) $f\cdot F$, as $f\to 0$, $F\to\infty$; (*b*) $f\cdot F$, as $f\to 0$, $F\to 0$;

(*c*) $f-F$, as $f\to\infty$, $F\to\infty$; (*d*) $\sqrt[F]{f}$, as $f\to\infty$, $F\to\infty$.

4. Discuss each of the following as a form assumed by an expression involving *two* variable quantities f and F, — one for each figure or symbol, — as was done for 0^∞ in Ex. III, p. 308:

(*a*) $\infty^{-\infty}$, (*b*) ∞^0, (*c*) ∞^1,

(*d*) $\infty+\infty$, (*e*) 2^∞, (*f*) $1^\infty+\frac{1}{\infty}$.

5. Find by inspection the limit of each of the following expressions as $x\to 0$:

(*a*) xe^{x^2}, (*b*) $\dfrac{x^2+4}{x+5}$, (*c*) $\dfrac{5}{x}$,

(*d*) $\dfrac{\cos x}{4+\sin x}$, (*e*) $\dfrac{x^2}{\sqrt{x+5}}$, (*f*) $\dfrac{\log x}{x}$.

6. Find by inspection the following limits:

(*a*) $\underset{x\to\frac{\pi}{2}}{\mathrm{L}}\dfrac{\tan x}{x}$, (*b*) $\underset{x\to\infty}{\mathrm{L}}\dfrac{\sin x}{x}$, (*c*) $\underset{x\to\infty}{\mathrm{L}}\dfrac{1000}{x\log x}$.

7. Explain why $\underset{x\to 0}{\mathrm{L}}\dfrac{\sin x}{x}$ is doubtful without study. Find its value by first writing the Maclaurin series for $\sin x$ and dividing by x. Also find this value by the differentiation method. If $y=(\sin x)/x$, what would you regard as the value of y when $x=0$?

8. The same as Ex. 7 for $\underset{x\to 0}{\mathrm{L}}\dfrac{e^{3x}-1}{x}$.

9. Which of the following limits are doubtful until investigated, and why? Give the actual values of the others.

(*a*) $\underset{x\to\frac{\pi}{2}}{\mathrm{L}}(\tan x)^{\cos x}$, (*b*) $\underset{x\to\frac{\pi}{4}}{\mathrm{L}}(\tan 2x)^x$,

(*c*) $\underset{k\to\infty}{\mathrm{L}}\left(1+\dfrac{r}{k}\right)^{kn}$, (*d*) $\underset{x\to 3}{\mathrm{L}}\dfrac{x^2-9}{x+1}$,

(*e*) $\underset{\Delta x\to 0}{\mathrm{L}}\dfrac{\log(x+\Delta x)-\log x}{\Delta x}$, (*f*) $\underset{x\to\frac{\pi}{2}}{\mathrm{L}}\tan x\sin 2x$.

10. By *Intro.*, §§ 163, 176, what are the values of the limits in Ex. 9 (*c*), (*e*), above?

11. Find the following limits by the general differentiation method:

(*a*) $\underset{x \to 0}{\mathrm{L}} \dfrac{\sin 4x}{x}$, (*b*) $\underset{y \to 0}{\mathrm{L}} \dfrac{e^{2y}-1}{\tan y}$, (*c*) $\underset{u \to 0}{\mathrm{L}} \dfrac{1-\cos u}{u^2}$,

(*d*) $\underset{x \to 1}{\mathrm{L}} \dfrac{x^2-1}{\sin \pi x}$, (*e*) $\underset{\theta \to \pi}{\mathrm{L}} \dfrac{\tan \theta}{\tan 2\theta}$, (*f*) $\underset{u \to 0}{\mathrm{L}} \dfrac{\sec 2u-1}{\cosh u-1}$,

(*g*) $\underset{x \to 2}{\mathrm{L}} \dfrac{\sqrt{x^3-8}}{x^2-4}$, (*h*) $\underset{\phi \to \frac{\pi}{2}}{\mathrm{L}} \dfrac{\operatorname{ctn} \phi}{2\phi-\pi}$, (*i*) $\underset{x \to 0}{\mathrm{L}} \dfrac{x \sin x}{e^x-1}$,

(*j*) $\underset{x \to 0}{\mathrm{L}} \dfrac{1-\cos x}{\sinh x}$, (*k*) $\underset{t \to 1}{\mathrm{L}} \dfrac{t^3-3t+2}{t^4-7t+6}$, (*l*) $\underset{y \to \infty}{\mathrm{L}} \dfrac{2\tan^{-1}(e^{-y})}{e^{-y}}$,

(*m*) $\underset{x \to 0}{\mathrm{L}} \dfrac{\sec x-1}{\tan^2 3x}$, (*n*) $\underset{n \to 2}{\mathrm{L}} \dfrac{n^3-12n+16}{n^3-3n^2+4}$, (*o*) $\underset{u \to 0}{\mathrm{L}} \dfrac{e^{u^2}-1}{3^{u^2}-1}$.

12. Show that the fraction in (37) approaches the limit log 2, as stated in § 181.

§ 184. The Form $\frac{\infty}{\infty}$. The differentiation method used for evaluating the form $0/0$ applies also to the form ∞/∞. But in all cases of ∞/∞ where x itself is finite, it is necessary, sooner or later, to convert to a form $0/0$ by some simplification or change of form.

Ex. I. Find the limit $V = \underset{x \to 0}{\mathrm{L}} \dfrac{\log x}{\operatorname{ctn} x}$.

At $x=0$, this takes the form ∞/∞. Differentiating separately gives

$$V = \underset{x \to 0}{\mathrm{L}} \frac{1/x}{-\csc^2 x}. \qquad \text{(Indeterminate)}$$

At $x=0$, this still takes the form ∞/∞. To escape repeating this, we note that $\csc^2 x$ is the reciprocal of $\sin^2 x$; and hence we write:

$$V = \underset{x \to 0}{\mathrm{L}} \frac{-\sin^2 x}{x}. \tag{41}$$

This is still indeterminate, but has the form $0/0$. Differentiating,

$$V = \underset{x \to 0}{\mathrm{L}} \frac{-2\sin x \cos x}{1} = 0.$$

(If we made the mistake of differentiating again, we should get the erroneous limit ∞.)

§ 185. The Forms $0 \cdot \infty$ and $\infty - \infty$. To find the limit of a product when one factor approaches zero while the other increases without limit, throw the product into a fractional form by writing

$$f \cdot F = \frac{F}{1/f}.$$

Thus, instead of the form $0 \cdot \infty$, we have to deal with a form $0/0$ or ∞/∞, which we handle as before. (§ 183.)

Likewise, to find the limit of a difference between two functions, each of which increases indefinitely, throw the difference into a fractional form $0/0$ or ∞/∞, and proceed as formerly. Or, as is sometimes possible, factor the difference so as to get a familiar product.

Forms like $0 \cdot \infty$ and $\infty - \infty$ must not be differentiated until a fractional form has been obtained, — of a standard type, *i.e.*, $0/0$ or ∞/∞. There must be no doubt that the numerator and denominator *both* approach zero, or *both* approach infinity.

Ex. I. Find the limit $L = \underset{x \to \infty}{\mathrm{L}}\, x^2 e^{-x}$.

Since $e^{-x} = \dfrac{1}{e^x}$, we find

$$L = \underset{x \to \infty}{\mathrm{L}} \frac{x^2}{e^x} = \underset{x \to \infty}{\mathrm{L}} \frac{2x}{e^x} = \underset{x \to \infty}{\mathrm{L}} \frac{2}{e^x} = 0.$$

Evidently *any other product of the form* $x^n e^{-x}$ would also approach the limit zero as $x \to \infty$.

Ex. II. Find the limit $L = \underset{x \to \infty}{\mathrm{L}} (e^x - x^2)$.

The given quantity can be factored thus:

$$e^x - x^2 = e^x\left(1 - \frac{x^2}{e^x}\right).$$

By Ex. I the second factor approaches 1. But the first factor becomes infinite. Hence the limit is infinite.

Or, the given quantity can be written as a fraction, viz.

$$\frac{1-x^2e^{-x}}{e^{-x}}.$$

By Ex. I again, the numerator approaches 1. But the denominator approaches 0; hence, the limit is infinite. To apply the differentiation method would be incorrect.

§ 186. The Forms ∞^0, 0^0, 1^∞. These arise from exponential forms such as $y = F^f$. But

$$\log y = f \log F;$$

and this is merely a product, which takes the form $0 \cdot \infty$ in each of the cases under consideration. If we can find the limit of $\log y$, the limit of y itself will be evident.

E.g., if $\log y \to 3$, then $y \to e^3$.

To evaluate the indeterminate product $f \cdot \log F$, we convert it into a fraction as heretofore.

Thus the evaluation of all our indeterminate forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 0^0, \quad 1^\infty, \tag{42}$$

goes back to the case of a fraction which takes the form $0/0$ or ∞/∞. And this we evaluate by differentiating numerator and denominator separately.

Ex. I. Find the limit $L = \underset{x \to 0}{\mathrm{L}} (\sin x)^{\sqrt{x}}$.

Putting $y = (\sin x)^{\sqrt{x}}$, we have

$$\log y = \sqrt{x} \log \sin x = \frac{\log \sin x}{1/\sqrt{x}}.$$

As $x \to 0$, this takes the form ∞/∞.

Differentiating once we find:

$$\underset{x \to 0}{\mathrm{L}} (\log y) = \underset{x \to 0}{\mathrm{L}} \frac{\operatorname{ctn} x}{-\frac{1}{2} x^{-\frac{3}{2}}} = \underset{x \to 0}{\mathrm{L}} \frac{-2\, x^{\frac{3}{2}}}{\tan x}.$$

Differentiating again, this reduces to

$$\underset{x \to 0}{\mathrm{L}} \frac{-3\sqrt{x}}{\sec^2 x} = 0.$$

Since $\log y \to 0$, therefore $y \to 1$. That is, $L = 1$.

EXERCISES

1. State again, briefly, *why* each of the forms in (42) is indeterminate.

2. Find each following limit. [Each independent variable approaches its own limit from above, if finite; and if it becomes infinite, does so positively. In each case, state whether reversing this mode of approach would affect the result.]

(*a*) $\underset{x \to 0}{\mathrm{L}} \dfrac{3x^2+5}{4x^2-7}$,

(*b*) $\underset{x \to \frac{\pi}{2}}{\mathrm{L}} \dfrac{\sec 3x}{\sec 5x}$,

(*c*) $\underset{x \to 0}{\mathrm{L}} \dfrac{\log x^2}{\operatorname{ctn} x}$,

(*d*) $\underset{y \to \infty}{\mathrm{L}} \dfrac{y^3}{e^y}$,

(*e*) $\underset{y \to 2}{\mathrm{L}} \dfrac{\tan\left(\frac{\pi y}{4}\right)}{\csc\left(\frac{\pi}{2}y\right)}$,

(*f*) $\underset{t \to 0}{\mathrm{L}} \left(\dfrac{e^t}{t} - \dfrac{5}{\sin t}\right)$,

(*g*) $\underset{t \to \infty}{\mathrm{L}} 3^{-t}t^3$,

(*h*) $\underset{x \to 2}{\mathrm{L}} x^2 \log x$,

(*i*) $\underset{\theta \to \frac{\pi}{4}}{\mathrm{L}} \tan 2\theta \cdot \tan 4\theta$,

(*j*) $\underset{x \to \infty}{\mathrm{L}} (e^x - x)$,

(*k*) $\underset{x \to \infty}{\mathrm{L}} (x - \sqrt{x^2-100})$,

(*l*) $\underset{u \to \infty}{\mathrm{L}} \tanh u$,

(*m*) $\underset{x \to 0}{\mathrm{L}} (\operatorname{ctn} x)^{\frac{1}{\log x}}$.

(*n*) $\underset{y \to 0}{\mathrm{L}} (e^y - 1)^{\sin y}$,

(*o*) $\underset{\phi \to \frac{\pi}{2}}{\mathrm{L}} (\sin \phi)^{\tan \phi}$,

(*p*) $\underset{\theta \to 0}{\mathrm{L}} (\sin \theta)^{\frac{1}{\log \theta}}$,

(*q*) $\underset{\phi \to 0}{\mathrm{L}} (1+\sin \phi)^{\operatorname{ctn} \phi}$

(*r*) $\underset{y \to 0}{\mathrm{L}} (1-\cos y)^{\frac{4}{\log y}}$,

(*s*) $\underset{y \to 0}{\mathrm{L}} (\sin e^{-\frac{1}{y}})^{2y}$,

(*t*) $\underset{\theta \to \frac{\pi}{2}}{\mathrm{L}} (1+\cos \theta)^{\tan 3\theta}$,

(u) $\underset{z \to 0}{\mathrm{L}} (e^z-1)^{\frac{\log 3}{\log z}}$, (v) $\underset{x \to \infty}{\mathrm{L}} (\tan e^{-x})^{\frac{1}{x}}$,

(w) $\underset{\phi \to \frac{\pi}{2}}{L} (\tan \phi)^{\operatorname{ctn} \phi}$, (x) $\underset{\phi \to \frac{\pi}{2}}{\mathrm{L}} (\tan \phi)^{\frac{1}{\log (2\phi-\pi)}}$.

3. Prove that the hyperbola $y^2=4x^2-25$ approaches the line $y=2x$ asymptotically as $x \to \infty$.

4. The same as Ex. 3 for the hyperbola $y^2=9x^2-36x$ and the line $y=3x-6$.

5. Plot $y = \tan 3x/\tan x$ from $x=60°$ to $x=90°$, at intervals of 10°.

6. Approximate $\int_0^{\frac{2\pi}{9}} \frac{\operatorname{ctn} 2\theta}{\operatorname{ctn} \theta} d\theta$ by Simpson's Rule for one interval.

7. Approximate each following integral by the series method, and also by Simpson's Rule using two intervals:

(a) $\int_0^{.4} \frac{\sin x}{x} dx$, (b) $\int_0^{.4} \frac{\log(1+x)}{x} dx$.

8. Show that the limit of x^5e^{-x} as $x \to \infty$ is zero; likewise the limit of x^pe^{-x}, if p is any real number.

9. Find $\int x^3e^{-x}dx$, omitting the constant of integration. Also find what limit the integral approaches as $x \to \infty$.

10. In the standard integration formula for a power,

$$\int_1^X x^n dx = \frac{X^{n+1}-1}{n+1},$$

find the limit of the right-hand member as $n \to -1$. Compare with the standard formula for the integral of $x^{-1}\,dx$.

11. Like Ex. 10 for $\int \sin mx \sin nx\, dx$, as $n \to m$.

§ 187. Improper Integrals: Type I. In various scientific problems we need the idea of a definite integral with an infinite upper limit. For such an integral the usual definition would be meaningless. We cannot set up a sum of n terms of the form (15), p. 94, which will reach to infinity. A new definition is needed, as follows:

By the integral of $f(x)dx$ from a to ∞, we shall understand

the limit approached by the corresponding integral from a to X, as $X \to \infty$:

$$\int_a^\infty f(x)dx = \underset{X\to\infty}{\mathrm{L}} \int_a^X f(x)dx. \tag{43}$$

If the integral from a to X fails to approach a finite limit as $X \to \infty$, we say that the integral from a to ∞ does not exist.

Ex. I. Find the integral $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$.

Here the integral from 1 to any value X is

$$\int_1^X \frac{dx}{1+x^2} = \arctan X - \arctan 1. \tag{44}$$

As $X \to \infty$, $\arctan X \to \dfrac{\pi}{2}$. But $\arctan 1 = \dfrac{\pi}{4}$. Hence

$$\int_1^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}. \tag{45}$$

Letting $X \to \infty$ here takes the place of substituting the upper limit directly in the indefinite integral, in an ordinary case. Notice that, if we here substitute ∞ for x, for brevity, getting $\arctan \infty - \dfrac{\pi}{4}$, no significance can be attached to this symbol $\arctan \infty$, nor any value attributed to it, except as the limit of $\arctan X$ when $X \to \infty$.

Geometrically speaking: The curve $y = 1/(1+x^2)$ approaches the X-axis asymptotically as $x \to \infty$. The curve and axis do not inclose an area at the right, — though only the value $y=0$ is permanently lacking. But there is a definite area under the curve from $x=1$ to any other point $x=X$. This area approaches a limit as $X \to \infty$; and that *limit* is regarded as the area under the curve from 1 to ∞.

FIG. 106.

An area "to infinity" cannot exist in even this limiting sense, unless $y \to 0$ when $x \to \infty$. And it may not exist then. (Cf. Ex. 1, p. 319.)

Ex. II. Calculate the value of the integral

$$P=\int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2}\,dx. \tag{46}$$

We proceed indirectly, as follows: An equal integral is

$$P=\int_{-\infty}^{\infty}e^{-\frac{1}{2}y^2}\,dy. \tag{47}$$

$$\therefore\ P^2=\left(\int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2}\,dx\right)\left(\int_{-\infty}^{\infty}e^{-\frac{1}{2}y^2}\,dy\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2}\cdot e^{-\frac{1}{2}y^2}dy\,dx.$$

This change of position is possible because each integral here is a constant. Further, combining exponents:

$$P^2=\int_0^{\infty}\int_0^{\infty}e^{-\frac{1}{2}(x^2+y^2)}\,dy\,dx. \tag{48}$$

Imagine now a surface $z=e^{-\frac{1}{2}(x^2+y^2)}$. The volume under it, over the entire XY-plane to infinity, would be precisely the double integral in (48). But this volume is easily found if we use cylindrical elements, $z\,r\,d\theta\,dr$:

$$V=\int_0^{\infty}\int_0^{2\pi}e^{-\frac{1}{2}r^2}r\,d\theta\,dr=2\pi\Big[-e^{-\frac{1}{2}r^2}\Big]_0^{\infty}=2\pi. \tag{49}$$

Since P^2 also equals V, therefore $P=\sqrt{2\pi}$.

(We have assumed here that the limit of the volume over a large square base would equal the limit over a large circular base, as each base "becomes infinite.")

One reason why infinite limits arise in scientific theory is that a given quantity or its influence may be known to extend exceedingly far out, whether without end or not. We can be sure of getting the entire quantity by integrating "to infinity." And in those cases any discrepancy due to using infinity instead of the true large unknown limit is negligible. An able physicist has said that, in his field, "a quantity is regarded as infinite when it is so large that making it any larger would have no appreciable effect."

§ 188. **The Gamma Function.** A very important type of integral with an infinite limit is

$$\int_0^{\infty}x^pe^{-x}dx. \tag{50}$$

When p is positive, integrating by parts with $u=x^p$ and $dv=e^{-x}\,dx$ replaces (50) by

$$-x^pe^{-x}\Big]_0^\infty + p\int_0^\infty x^{p-1}e^{-x}\,dx. \tag{51}$$

As $x \to \infty$, the term $-x^pe^{-x}$ becomes indeterminate; but it approaches the limit zero. (Cf. Ex. 8, p. 315.) The same term vanishes when $x=0$. Hence we have the reduction formula:

$$\int_0^\infty x^pe^{-x}\,dx = p\int_0^\infty x^{p-1}e^{-x}\,dx. \tag{52}$$

If p is an integer, repeated application of (52) will bring us to a known integral. *E.g.*,

$$\int_0^\infty x^4e^{-x}\,dx = 4\cdot 3\cdot 2\cdot 1\cdot \int_0^\infty e^{-x}\,dx = 4! \tag{53}$$

But if p is not an integer, some power of x will always remain after using (52). Thus

$$\int_0^\infty x^{3.7}e^{-x}\,dx = (3.7)(2.7)(1.7)\int_0^\infty x^{.7}e^{-x}\,dx. \tag{54}$$

Notice, however, that we now have an integral in which the exponent p is less than 1.

The integral in (50) is called the Gamma Function of $(p+1)$, written $\Gamma(p+1)$; *i.e.*,

$$\Gamma(p+1) = \int_0^\infty x^pe^{-x}dx. \tag{55}$$

[The extra unit in the notation $p+1$ is an unfortunate survival of the historical development of the theory. In any given case, fix attention on the actual exponent in the integrand; that exponent is called p.]

For fractional values of p between 0 and 1, tables of $\Gamma(p+1)$ have been calculated, — by methods explained in more advanced courses. And, from what was said above, it is clear that the Gamma Function for a larger value of p can be reduced until the exponent is less than 1. In fact, (52) may be rewritten

$$\Gamma(p+1) = p\,\Gamma(p). \tag{56}$$

The following examples will show the use of the small table on p. 501; and a few illustrations of the application of gamma functions are given in the exercises.*

What we have said has concerned only the case where p is positive. In higher courses, it is shown that the integral (55), *i.e.*, $\Gamma(p+1)$, does not exist unless p lies above -1.

Ex. I. Find $\int_0^\infty x^{.6}e^{-x}\,dx$. [That is, $\Gamma(1.6)$.]

For $p=.6$ (the actual exponent in the integrand), we read from the table: $\Gamma(1.6)=.8935$.

Ex. II. Find $\Gamma(3.6)$.

By (56): $\Gamma(3.6)=(2.6)\Gamma(2.6)=(2.6)(1.6)\Gamma(1.6)$.

Or, since we have found $\Gamma(1.6)$ to be .8935, this gives

$$\Gamma(3.6)=(2.6)(1.6)(.8935)=3.7170.$$

EXERCISES

1. Draw each following curve from $x=1$ to $x=9$; also determine whether an area to infinity will exist:

(a) $y=\dfrac{1}{x}$, (b) $y=\dfrac{1}{x^2}$, (c) $y=\dfrac{1}{\sqrt{x}}$.

2. With the help of the tables find the following:

(a) $\int_0^\infty x^{.25}e^{-x}\,dx$, (b) $\int_0^\infty y^{.15}e^{-y}\,dy$, (c) $\Gamma(1.75)$,

(d) $\int_0^\infty x^{2.4}e^{-x}\,dx$, (e) $\int_0^\infty t^{3.2}e^{-t}\,dt$, (f) $\Gamma(2.7)$,

(g) $\int_0^\infty x^{.78}e^{-x}\,dx$, (h) $\Gamma(1.42)$, (i) $\Gamma(3\tfrac{2}{3})$.

3. Find graphically the integral: $\int_0^9 x^{\frac{1}{2}}e^{-x}\,dx$. Compare with the value of $\Gamma(1.5)$.

4. Find $\Gamma(.3)$ or $\int_0^\infty x^{-.7}e^{-x}\,dx$. [Hint: We can look up $\Gamma(1.3)$; and, by (56), we know that $\Gamma(1.3)=.3\Gamma(.3)$. Hence we can find $\Gamma(.3)$].

5. The same as Ex. 4 for $\Gamma(.8)$ or $\int_0^\infty x^{-.2}e^{-x}\,dx$.

* For a large table see J. W. Glover, *Tables of Applied Mathematics in Finance, Insurance, Statistics.*

6. The median ordinate in a certain type of curve fitted to statistical data is found by the formula

$$y_0=\frac{N}{a(2)^{2m+1}}\,\frac{\Gamma(2\,m+2)}{[\Gamma(m+1)]^2}. \tag{57}$$

If $N=2000$, $m=4.25$, and $a=5$, find y_0. (Use logarithms.)

7. The formula corresponding to (57) for another curve is

$$y_0=\frac{Ne^p\,p^{p+1}}{a\,\Gamma(p+1)}. \tag{58}$$

Find y_0 if $N=1200$, $a=.1$, and $p=.08$.

8. Evaluate the following integrals:

(a) $\displaystyle\int_2^\infty\left(\frac{x}{\sqrt{x^2-1}}-\frac{x^2}{x^2+1}\right)dx,$ (b) $\displaystyle\int_1^\infty\frac{1-\log x}{x^2}\,dx,$

(c) $\displaystyle\int_1^\infty\left(2\,\mathrm{ctn}^{-1}\,x-\frac{2\,x}{x^2+1}\right)dx,$ (d) $\displaystyle\int_0^\infty\frac{dx}{\sqrt{x^2+1}}.$

9. The total energy of the magnetic field created by a certain moving electrical charge is expressible as

$$E=\frac{e^2v^2}{8\,\pi}\int_a^\infty\int_0^\pi\int_0^{2\pi}\frac{\sin^3\phi}{r^2}\,d\theta\,d\phi\,dr.$$

Evaluate this triple integral and simplify the formula.

10. The velocity of effusion of a gas through a small opening is

$$V=a\int_0^\infty e^{-ku^2}u\,du,$$

where a and k are constants. Evaluate and simplify.

11. The electrostatic potential at a point A is the work necessary to bring a unit positive charge from an infinite distance to A. Find the potential 5 cm. from a charge of E units. (The repulsive force exerted by E upon the approaching unit charge, at any distance x, will be E/x^2.)

12. Find the total attraction exerted upon a particle of mass m located a cm. from a flat plate, of a constant surface density k, and of indefinitely great extent. (Cf. § 120.)

13. Evaluate the following integrals:

(a) $\displaystyle\int_0^\infty x^3\,e^{-\frac{x}{4}}\,dx,$ (b) $\displaystyle\int_0^\infty x^2\,e^{-\frac{x}{a}}\,dx.$

[Hint: Put the exponent of e equal to $-y$, and complete the substitution thus begun.]

14. For all particles of a gas, the mean value of the square of the free-path length x is

$$\frac{1}{l}\int_0^\infty x^2\,e^{-\frac{x}{l}}\,dx,$$

where l is a constant. Show that this mean value is $2\,l^2$.

15. The coefficient of viscosity of a fluid (η) is related thus to the "mean momentum transfer distance" (L) of migrating molecules:

$$\eta = \frac{k}{L^2}\int_0^\infty \int_0^{\frac{\pi}{2}} z^2\, e^{-\frac{z}{L}} \sin\theta \cos\theta \, d\theta \, dz.$$

Show that this reduces to $\eta \doteq kL$.

16. In the integral $\int_0^\infty e^{-\frac{1}{2}x^2}\,dx$ put $\frac{1}{2}x^2 = t$, and complete the substitution. From this result and Ex. II, § 187, calculate $\Gamma(\frac{1}{2})$. Check by the fact that $\Gamma(1.5) = \frac{1}{2}\Gamma(\frac{1}{2})$.

§ 189. Normal Law of Error. In making any measurement, there is likely to be some error; this may be positive or negative, large or small. Experience shows, however, that large errors are much less common than small errors. (Cf. *Intro.*, §§ 339–341.)

For any one kind of measurement, let P denote the probability of an error between 0 and x, — in other words, the percentage or fraction of errors that would ultimately fall between 0 and x. Then ΔP denotes the ultimate percentage falling in an interval Δx, between x and $x + \Delta x$. The average percentage per unit range of error in the interval Δx is $\Delta P/\Delta x$. The limit of $\Delta P/\Delta x$, as $\Delta x \to 0$, we shall call simply the "percentage per unit," at that x value. Denoting this by y:

$$\frac{dP}{dx} = y. \qquad \therefore\ P = \int y\,dx. \tag{59}$$

Hence, if a graph were plotted, with the error x as abscissa, and with the percentage per unit, y, as ordinate, its area from $x = 0$ to $x = X$ would numerically equal the ultimate percentage of errors falling between 0 and X, — or the probability of an error in that interval. Likewise the area from $x = x_1$ to $x = x_2$ would equal the probability of an error between x_1 and x_2. (Fig. 107.)

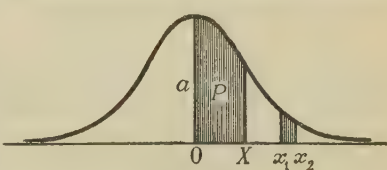


FIG. 107.

On the basis of apparently reasonable assumptions, it is shown in texts on the Theory of Errors that

$$y = ae^{-\pi a^2 x^2}, \tag{60}$$

where a is a constant, viz. the percentage per unit at $x=0$. Then, by (59), the percentage between 0 and X is

$$P(X) = \int_0^X ae^{-\pi a^2 x^2}\,dx. \tag{61}$$

This is called the Normal Probability Integral. The equation (60) is called the Normal Law of Error, and its graph the Normal Probability Curve. This curve runs to infinity, approaching the X-axis asymptotically; but it is soon so close to the axis as to leave virtually no chance for an error x beyond some moderate value.

A check on the foregoing relations is this: The integral (61), with ∞ as its upper limit, turns out to be $\frac{1}{2}$, — which is what we should expect as the probability that an error be positive.

From the meaning of a, we see that its value depends upon the degree of precision attainable in the particular kind of measurement considered. For high accuracy in comparison with the size of the chosen unit of measure, a is large. The graph then is high at $x=0$ and falls very rapidly on each side. For low accuracy, a is small; the graph is low at $x=0$, and falls slowly. Figure 107 shows that shape when $a=1$. Any value of a can be used by choosing suitably the unit in terms of which the error x is stated.

Sometimes it is convenient to use different scales for x and y in plotting, which steepens or flattens the curve. In effect, this replaces a true normal curve by another curve, whose height is everywhere some constant c times the true height. The total area, from $-\infty$ to $+\infty$, will be c instead of 1. Likewise the area from x_1 to x_2 will be c times the true area; but if divided by c (*i.e.*, by the entire area), the quotient or percentage will still give the correct probability of an error between x_1 and x_2. (Cf. Ex. I below.)

The equation of the curve as actually plotted would be

$$y = cae^{-\pi a^2 x^2}, \tag{62}$$

or, if we call $ca = b$ and $\pi a^2 = n$:

$$y = be^{-nx^2}. \tag{63}$$

Various other forms are possible.

All that we have said concerning errors of measurement applies also to *deviations from an average size or value*, in the case of many biological and economic variables, though a normal probability distribution for such deviations is by no means universal.

Tables. Several tables are available which give the value of the integral in (61), that is, of P, for various values of the upper limit X. Some tables employ one value of a, and others a different value. For statistical studies it is most convenient to have $a = 1/\sqrt{2\pi}$. Then (60) and (61) become

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \tag{64}$$

$$P(X) = \frac{1}{\sqrt{2\pi}} \int_0^X e^{-\frac{1}{2}x^2}\, dx. \tag{65}$$

The small table on p. 501 is given for this form.* For other forms of the probability formula, the integrals are reducible to this form.

Ex. I. Find the probability of an error between .8 and 1.6 units, when the law of error has the standard form (64).

The table, p. 501, gives immediately the probability from 0 to 1.6 and that from 0 to .8, respectively, as

$$P(1.6) = .4452, \qquad P(.8) = .2881.$$

Subtracting gives the required probability as .1571.

Ex. II. Find the ultimate percentage p of certain biological deviations falling between .2 and .4 if the equation of the normal curve as plotted for the data is $y = 3\, e^{-8x^2}$.

* See § 180, also J. W. Glover: *Tables of Applied Mathematics in Finance, Insurance, Statistics.*

Here p equals the area between .2 and .4, divided by the total area:

$$p=\int_{.2}^{.4} 3\,e^{-8x^2}\,dx \div \int_{-\infty}^{\infty} 3\,e^{-8x^2}\,dx. \tag{66}$$

Let $8\,x^2=\frac{1}{2}\,t^2$. Then $x=\frac{1}{4}\,t$, $dx=\frac{1}{4}\,dt$. Also $t=.8$ when $x=.2$, and $t=1.6$ when $x=.4$.

$$\therefore \qquad p=\tfrac{3}{4}\int_{.8}^{1.6} e^{-\frac{1}{2}t^2}\,dt \div \tfrac{3}{4}\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}\,dt. \tag{67}$$

By Ex. II, p. 317, the latter integral equals $\sqrt{2\,\pi}$

$$\therefore \qquad p=\frac{1}{\sqrt{2\,\pi}}\int_{.8}^{1.6} e^{-\frac{1}{2}t^2}\,dt. \tag{68}$$

This last is the standard form for our tables; and in Ex. I was found to be $p=.1571$. For the problem in question, about 15.71% of all deviations in the long run should fall between .2 and .4 units.

§ 190. Improper Integrals: Type II. We have seen that a special definition is required for an integral if it has an infinite limit. The same is true if the integrand becomes infinite within the interval of integration. For, in defining

$$\int_a^b f(x)\,dx \tag{69}$$

in §§ 58–59, we stipulated that $f(x)$ should be continuous from $x=a$ to $x=b$. Both the definition and the method of calculating the integral by forming the difference of two values of the integral function, $F(b)-F(a)$, were based upon the assumption of continuity.

If $f(x)$ becomes infinite or is otherwise discontinuous at a single value between a and b, say at $x=c$, we define the integral (69) as follows:

Let $T(h)$ denote the sum of the two integrals

$$\int_a^{c-h} f(x)dx \qquad \text{and} \qquad \int_{c+h}^{b} f(x)dx.$$

Then, if $T(h)$ approaches a limit as $h \to 0$, we call that limit the integral of $f(x)\,dx$ from a to b. If no limit is approached, we say that the integral from a to b does not exist.

Geometrically speaking: If the sum of the two shaded areas in Fig. 108 approaches a limit as their dotted boundaries are brought together, we call that limit the area under the curve from a to b. No area is bounded in one sense, inasmuch as there is no point where the two parts of the curve come together; but there may be an area in the limit sense. And, again, there may not: all depends upon how fast the curve rises as $x \to c$.

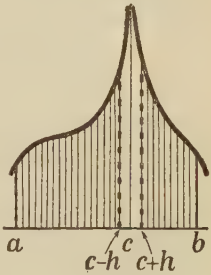


FIG. 108.

If $f(x)$ is discontinuous at one *end*, say at $x=b$, the definition of the integral (69) is similar, but $T(h)$ is taken as the integral from a to $b-h$. If $f(x)$ is discontinuous at several values c_1, c_2, c_3, etc., we again proceed likewise, taking $T(h)$ as the sum of such integrals within all the intervals concerned: a to c_1-h, c_1+h to c_2-h, etc.

Some integrals already used have been of this "improper" type, in which the integrand becomes infinite. *E.g.*, we have written

$$\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}\Big]_0^a = \frac{\pi}{2}. \tag{70}$$

Here the integrand is infinite at $x=a$. But the function $T(h)$ mentioned above would be

$$T(h) = \int_0^{a-h} \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{a-h}{a}, \tag{71}$$

and clearly this approaches $\pi/2$ as $h \to 0$.

Similar cases of improper integrals already encountered have been such as to involve no danger of error. But it is important to grasp this final caution: It is not safe merely to find an indefinite integral and take the difference of two of its values, $F(b)-F(a)$, without considering the nature of the integrand and the limits.

Ex. I. Examine the area under the curve $y=\dfrac{1}{(x-3)^2}$ between $x=1$ and $x=5$.

Using the ordinary formula blindly:

$$A=\int_1^5 y\,dx=-\frac{1}{x-3}\bigg]_1^5=-\frac{1}{2}-\left(\frac{1}{2}\right)=-1. \qquad (72)$$

But this is ludicrously incorrect. Note the discontinuity of y at $x=3$, and form the function $T(h)$:

$$T(h)=\int_1^{3-h}\frac{dx}{(x-3)^2}+\int_{3+h}^5\frac{dx}{(x-3)^2}=\left(\frac{1}{h}-\frac{1}{2}\right)+\left(-\frac{1}{2}+\frac{1}{h}\right). \qquad (73)$$

As $h\to 0$, $T(h)\to\infty$. The curve not merely goes "to infinity" at $x=3$, but goes so rapidly that no area is bounded, even in the limit sense.

§ **191. Remarks on Chapter VII.** There are several outstanding points covered in the present chapter which should be kept always in mind:

The mean value of a varying quantity Q, defined as the limit of the arithmetical average of n suitably chosen values, as $n\to\infty$, can be found by integration. Symbolically

$$\bar{Q}=\frac{\int Q\,dR}{\int dR},$$

but several integrals may be required, according to the nature of the region R.

Definite integrals can be approximated graphically, by Simpson's Rule, by series, and sometimes in other ways. All of these methods should be so familiar as to occur to us at once in dealing with any troublesome integral.

Forms are not indeterminate unless there is some conflict of tendencies which makes an investigation necessary before the limit is known. Also, the one basic indeterminate form to which all others (with a single exception) *must be reduced,*

sooner or later, if the differentiation method is to be used, is the fraction 0/0. The exception is that a form ∞/∞, resulting from x itself becoming infinite, may sometimes be handled directly. And we note again that the numerator and denominator are to be differentiated *separately*.

The general form of each type of elliptic integral, and the gamma and probability integrals, should be sufficiently familiar to be recognized when seen, and to be found readily from a table.

And, finally, the limit method of defining and testing improper integrals should be thought of whenever there is an infinite limit of integration, or an integrand which is discontinuous anywhere in the interval.

EXERCISES

1. Find the value of each following integral:

(*a*) $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{1.5} e^{-\frac{1}{2}x^2}\,dx,$ (*b*) $\displaystyle\int_0^2 e^{-\frac{1}{2}x^2}\,dx,$ (*c*) $\displaystyle\int_{.3}^{.5} e^{-\frac{1}{2}x^2}\,dx.$

2. Find without tables: $\displaystyle\int_0^2 e^{-x^2}x\,dx,$ also $\displaystyle\int_0^\infty e^{-x^2}x\,dx.$

3. Reduce each following integral to the standard form for $P(X)$, aside from a constant factor; and evaluate. [Hint: Put the exponent of e equal to $-\frac{1}{2}t^2$.]

(*a*) $\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^1 e^{-2x^2}\,dx,$ (*b*) $\dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^{\sqrt{2}} e^{-x^2}\,dx,$ (*c*) $\sqrt{\pi}\displaystyle\int_0^5 e^{-.18x^2}\,dx.$

4. By integrating by parts reduce $\displaystyle\int_0^1 x^2 e^{-\frac{1}{2}x^2}\,dx$ to a standard $P(X)$ form.

5. Reduce the following to gamma functions by putting the exponent of e equal to $-t$; also try integration by parts:

(*a*) $\displaystyle\int_0^\infty x^3 e^{-\frac{1}{2}x^2}\,dx,$ (*b*) $\displaystyle\int_0^\infty x^2 e^{-x^2}\,dx.$

6. If the units employed in a measurement are such that the law of error has the standard form (64), find the probability of an error in each following interval:

(*a*) between 0 and 1.5; (*b*) between 1.5 and 2;
(*c*) between -1 and 1; (*d*) between -3 and 3.

7. A biological deviation x is distributed according to the equation $y=5\,e^{-4.5x^2}$. Find the ultimate percentage of deviations falling between .5 and 1.

8. In the long run, one-fourth of the errors of a measurement fall between zero and a value $x=E$, called "the probable error." Find E from the table, for the case of equation (64). What percentage of errors will fall between $x=E$ and $x=2\,E$? (Cf. *Intro.* § 341.)

9. A vast number (N) of positive errors are distributed according to (64). (*a*) Express the number that should fall in a tiny interval from x to $x+dx$. [Think of the area of the normal curve in that interval.] (*b*) Show that the mean error for the set of N is theoretically: $\bar{x}=\dfrac{1}{\sqrt{2\,\pi}}\displaystyle\int_0^\infty x\,e^{-\frac{1}{2}x^2}\,dx.$ Evaluate this expression.

10. In a gas with no mass motion the number of molecules having any velocity v [*i.e.*, between v and $v+dv$] is $KNe^{-kv^2}\,v^2\,dv$, where N is the total number of molecules, and K and k are constants. Express by an integral the number of molecules having a speed greater than a given value V.

11. (*a*) In Ex. 10 express the mean velocity $\bar{v}$ for all N molecules. (*b*) Reduce this to $\bar{v}=K\,\Gamma\,(2)/2\,k^2$, or $K/2\,k^2$.

12. In (60) put $x=0$, and thus verify the meaning attributed to a.

13. Examine the area under each of the following curves from $x=3$ to $x=12$:

(*a*) $y=\dfrac{1}{x-4}$, (*b*) $y=\dfrac{1}{(x-4)^2}$, (*c*) $y=\dfrac{1}{\sqrt[3]{(x-4)^2}}$.

14. Using polar coördinates calculate the shaded sectorial area of Fig. 97, p. 265, within any central angle $2\,\theta$. Do the curve and its asymptotes "bound" such a sectorial area in the limit sense?

15. One of Fresnel's integrals, very important in studying diffraction, is $\displaystyle\int_0^V \cos\left(\frac{\pi}{2}v^2\right)dv$. Calculate this approximately if $V=.2$.

16. Calculate, and check by tables: $\displaystyle\int_0^{\frac{\pi}{18}}\frac{d\theta}{\sqrt{1-.09\sin^2\theta}}$.

17. A point oscillates thus: $x=10\sin 50\,\pi t$. Find its mean speed during the first quarter-cycle. (Use time as the basis.)

18. The disturbing power of the sun on the orbit of the moon, when the earth is r units distant from the sun, is, very approximately: $D=K/r^3$, where K is a constant. Find $\bar{D}$ for all times during a revolution of the earth. [See § 174.]

19. The density of the earth's atmosphere (D tons per cu. mi.) varies approximately as follows with the distance (ρ mi.) from the center of the earth: $D = 5.28 \times 10^{350}\, e^{-.2\rho}$. Calculate the entire mass of the atmosphere, from $\rho = 3960$ on out.

20. The temperature ($T°$) of molten lava, at the center of a mass cooling under certain conditions, would be (after t sec.) approximately,

$$T = \frac{2000}{\sqrt{2\pi}} \int_0^{\frac{13000}{\sqrt{t}}} e^{-\frac{1}{2}y^2}\, dy. \tag{74}$$

Find T at the end of 1 day ($t = 86400$). Also at the end of 100 yr.

21. Find the following limits:

(a) $\underset{x \to 0}{\mathrm{L}} \dfrac{\log(1+x)}{e^{2x}-1}$, (b) $\underset{x \to 0}{\mathrm{L}} \dfrac{3^{2x}-3}{3^x-1}$, (c) $\underset{n \to \infty}{\mathrm{L}} \left(1 - \dfrac{1}{n}\right)^n$.

22. What is, or what had we best regard as, the value of y at $x = 3$ in each following case?

(a) $y = \dfrac{e^x - e^3}{x-3}$, (b) $y = \dfrac{x^3 - 27}{x+1}$,

(c) $y = \dfrac{\log x - \log 3}{x^2 - 9}$, (d) $y = \dfrac{\cos(\pi/x)}{x}$,

(e) $y = \dfrac{\tan(\pi x/6)}{\sec(\pi x/2)}$, (f) $y = \dfrac{\log(x-3)}{1000000\, x}$.

23. The quantity of electricity discharged from a condenser in time t under certain conditions is $Q = C_1\, e^{-200t} + C_2 e^{rt}$, where r is a constant, and

$$C_1 = \frac{1100}{r+200}, \qquad C_2 = \frac{5r - 100}{r+200}. \tag{75}$$

If r be made to approach -200, show that the limiting form for Q is

$$Q = e^{-200t}\,[5 - 1100\, t]. \tag{76}$$

24. An integral encountered in the study of population growth has the form

$$Q = \int_0^\infty e^{-kx} f(x)\, dx. \tag{77}$$

Denote by F_n the similar integral, $\int_0^\infty x^n f(x) dx$, and express Q as a series in terms of F_0, F_1, F_2, etc.

25. Find the total attraction upon a particle P, of mass m, located a cm. from a thin straight wire, of linear density k, which extends a very great distance in each direction from its point O nearest to P.

26. Like Ex. 25, if P is in line with the exceedingly long wire and a cm. from its nearer end.

CHAPTER VIII

DIFFERENTIAL EQUATIONS

§ 192. How Differential Equations Arise. Often our only information connecting two variables x and y is some known fact concerning their rates of change, — expressible by an equation involving derivatives or differentials. Such an equation is called a " differential equation." We shall often abbreviate this " *D.E.*"

To " solve " a *D.E.* is to find the relation between the variables x and y themselves, free from derivatives. The method depends on the *form* of the *D.E.*, — especially the order and degree of the highest derivative present.

A second derivative like d^2y/dx^2 is of the "second order"; any derivative squared is of the "second degree."

Before studying various methods of solution, let us see in a few cases how a *D.E.* is set up originally.

Ex. I. A vertical spring, with its lower end fixed, is stretched slightly upward and released. It contracts to a length less than normal, then lengthens beyond normal, again contracts, and so on. Ignoring resistances, what is the *D.E.* for the motion of the free end P?

It is known from Physics that the restoring force at any instant is proportional to the elongation or contraction (y) then existing. Moreover, the acceleration of P is proportional to the force. Hence

$$\frac{d^2y}{dt^2}=ky. \tag{1}$$

This is the required *D.E.*; but let us make sure whether the constant k should have a positive or negative value.

When P is above E and moving upward, the speed dy/dt is positive but decreasing, which makes d^2y/dt^2 negative. When P is above E but moving downward, the speed is negative and increasing numerically, — hence still decreasing algebraically. Thus d^2y/dt^2 is negative wherever y is positive. The reverse is also true. Hence k must be negative in (1).

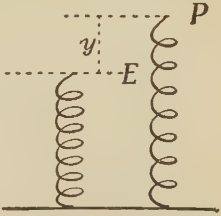


Fig. 109.

Ex. II. A rope is wound around a circular post and pulled tight. Due to friction the tension T in the wound portion diminishes as the distance x from the free end increases. (Fig. 110.) What is the $D.E.$ for this variation, if the rope is about to slip without stretching?

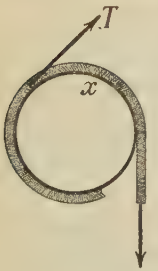


Fig. 110.

The rate of decrease of T per unit length equals the friction per unit length, at any point. This is proportional to the pressure against the post, and the pressure is proportional to the tension.

$$\therefore \qquad \frac{dT}{dx} = -kT. \tag{2}$$

Note. Neither of the $D.E.$'s (1) and (2) above can be solved by ordinary direct integration. For, each left member is a derivative with respect to t or x, while each right member is a function of y or T.

§ 193. Nature of a Solution. A solution of a differential equation is any relation $f(x, y) = 0$ which, together with derivatives calculated from it, will satisfy the $D.E.$ identically if substituted therein.

That is to say: all terms obtained on substituting must cancel out, without assigning special values to any variables which appear.

A solution may involve *arbitrary constants*, which can have any values as far as the $D.E.$ is concerned, but which become definite when initial values of x, y, dy/dx, etc., are known.

The number of arbitrary constants in the most general solution is equal to the order of the $D.E.$*

Ex. I. Determine whether the relation

$$y = Ae^{2x} + Be^{-5x} - \tfrac{1}{5}x - \tfrac{3}{50} \tag{3}$$

is a solution of the $D.E.$:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 2x. \tag{4}$$

We differentiate (3) twice, and substitute the resulting values of dy/dx and d^2y/dx^2 in the left member of (4), — which then becomes

$$(4Ae^{2x} + 25Be^{-5x}) + 3(2Ae^{2x} - 5Be^{-5x} - \tfrac{1}{5}) \\ - 10(Ae^{2x} + Be^{-5x} - \tfrac{1}{5}x - \tfrac{3}{50}).$$

No matter what values A and B may have, this reduces to $2x$, the right member of (4). Hence (3) is a solution of (4).

Moreover (3) is the general solution since it involves two arbitrary constants A and B.

If we were to test similarly the expression $y = Ae^{2x} + Be^{-5x} + Cx + D$, we should find it to be a solution of (4), *provided* $C = -\frac{1}{5}$ and $D = -\frac{3}{50}$. There would still be but two arbitrary constants A and B.

EXERCISES

1. Verify that each supposed solution here is actually a solution of the given differential equation. Is it the general solution?

	D.E.	*Solution?*
(*a*)	$d^2y/dx^2 - 7\,dy/dx + 12y = 0,$	$y = Ae^{3x} + Be^{4x}.$
(*b*)	$d^2y/dx^2 + 4y = e^{3x},$	$y = A\sin 2x + B\cos 2x + \frac{1}{13}e^{3x}.$
(*c*)	$d^3Q/dt^3 + 8Q = 0,$	$Q = Ae^{-2t} + e^t \sin\sqrt{3}\,t.$
(*d*)	$4\,d^2y/dx^2 = -[1 + (dy/dx)^2]^{\frac{3}{2}},$	$x^2 + y^2 = 16.$

2. Show that $y = x^{-2} + Cx^2$ is a solution of $dy/dx + 2y/x = x$, provided C has a certain value. What value?

3. The same as Ex. 2 for each following equation and $D.E.$:

(*a*) $y = A\sin 3x + B\cos 3x + C\sin x,$ $\quad d^2y/dx^2 + 9y = \sin x;$

(*b*) $y = Ae^{-3x} + Be^x + Cxe^x,$ $\quad d^2y/dx^2 + 2\,dy/dx - 3y = e^x.$

* For a proof see W. F. Osgood, *Advanced Calculus*, pp. 345–49.

4. In Fig. 109, why must d^2y/dt^2 be $+$ when y is $-$, throughout?

5. Write a *D.E.* expressing each of the following facts:

(*a*) An electric current dies out at a rate proportional to the intensity remaining;

(*b*) A scent followed by a dog fades away proportionally, as in (*a*);

(*c*) A boat gliding slowly is retarded, proportionally to the speed;

(*d*) The surface tension T of a serum, if suddenly lowered, increases for some time, — at a rate proportional to T.

6. Express by a *D.E.* the law of increase or decrease of each following quantity:

(*a*) The number of bacteria N in a culture, with ample food. [Each bacterium in turn becomes two, at regular intervals.]

(*b*) The quantity Q remaining from an original amount of radium. [Decomposition occurs continually as atoms explode in turn.]

(*c*) The atmospheric pressure (p lb./sq. ft.), with the elevation (h ft.), assuming the temperature constant. [What causes the difference dp in pressure between elevations h and $h+dh$? The weight of a tiny portion (of height dh) of a column of 1 sq. ft. base, is proportional to p and to dh.]

7. When a meteor falls directly toward the earth, how does the acceleration imparted to it by gravity vary with the distance from the center of the earth? Express this fact by a *D.E.*

8. Find the *D.E.* for the curve of the hollow upper surface of a fluid, which rotates at a constant angular speed ω. [Any surface particle P is in equilibrium. Hence the gravitational acceleration g and the centrifugal acceleration c must have a resultant perpendicular to the curve or its tangent line. Also, $c = \omega^2 x$. (Fig. 111.)]

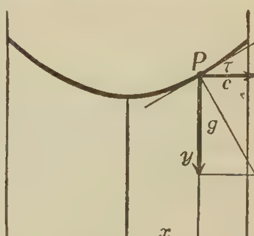


FIG. 111.

9. The normal probability curve may be defined geometrically as a curve whose slope at any point is $dy/dx = -xy$, and whose height at $x = 0$ is $1/\sqrt{2\pi}$. Derive equation (64), p. 323, from this *D.E.* [Hint. Divide through by y before integrating. Cf. *Intro.*, § 181.]

10. Due to leakage in an electric power transmission line, the current (I amp.) varies with the distance (x mi.) at a rate proportional to the e.m.f. (E volts). Also E varies at a rate proportional to I. Show that d^2I/dx^2 is proportional to I. How about d^2E/dx^2?

EQUATIONS OF THE FIRST ORDER

We come now to methods of solution; and begin with differential equations of the first order and first degree. Several varieties will be considered.

§ 194. Variables Separable. Multiplying by dx and other factors will sometimes reduce a *D.E.* to the form

$$M\,dx + N\,dy = 0, \tag{5}$$

where M is a function of x alone, and N of y alone. The variables x and y are then said to be "separated."

$$\therefore \qquad \int M\,dx + \int N\,dy = c. \tag{6}$$

This gives the relation between x and y themselves.

The *D.E.* is now regarded as solved, even if we do not recognize the integrals in (6). We can at least integrate approximately.

Some *D.E.*'s whose variables are not separable, as above, can be reduced to a separable case by substituting

$$y = vx, \tag{7}$$

$$dy = v\,dx + x\,dv. \tag{8}$$

This always works when a given *D.E.* is *homogeneous;* that is, when the coefficients of dx and dy involve only algebraic terms of some one degree in x and y, or transcendental terms which are unaltered by replacing x by kx and y by ky. (Cf. Ex. II below.)

Ex. I. $$\sqrt{1-x^2}\frac{dy}{dx} + \sqrt{1-y^2} = 0. \tag{9}$$

Multiplying through by dx, dividing by the product of the radicals, and reversing the order:

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} = 0. \tag{10}$$

The variables are separated. The solution is

$$\sin^{-1} x + \sin^{-1} y = c. \tag{11}$$

This can also be written in another form. (Ex. 13, p. 339.)

Ex. II. $$(4\,x^3+3\,xy^2)dx+(x^2y+y^3)dy=0. \tag{12}$$

This *D.E.* is homogeneous, every term being of degree 3 in x and y.

Substituting $y=vx$, $dy=v\,dx+x\,dv$:

$$(4\,x^3+3\,x^3v^2)dx+(x^3v+x^3v^3)(vdx+xdv)=0. \tag{13}$$

Factoring out x^3, collecting dx terms, and separating:

$$(4+4\,v^2+v^4)dx+(v+v^3)x\,dv=0, \tag{14}$$

$$\frac{dx}{x}+\frac{(1+v^2)v\,dv}{(2+v^2)^2}=0. \tag{15}$$

This can be integrated by partial fractions. But we may simplify by putting $2+v^2=t$:

$$\frac{dx}{x}+\frac{\frac{1}{2}(t-1)dt}{t^2}=0. \tag{16}$$

Multiplying by 2, integrating, and going back to v:

$$2\log x + \log(2+v^2)+\frac{1}{2+v^2}=\log k. \tag{17}$$

The constant of integration is called *log k* instead of c, in order to combine it more readily with the other logarithms present. Any constant c is the logarithm of some other constant k.

$$\therefore \qquad \log\frac{x^2(2+v^2)}{k}=-\frac{1}{2+v^2}, \tag{18}$$

i.e., $$x^2(2+v^2)=k\,e^{-\frac{1}{2+v^2}}. \tag{19}$$

Or, since $v=y/x$:

$$\therefore \qquad 2\,x^2+y^2=k\,e^{-\frac{x^2}{2x^2+y^2}}. \tag{20}$$

This is the required solution. [Cf. Ex. 11, p. 76.]

§ 195. **Biological Growth.** The growth of human infants, — and of various other young animals and some plants, —

proceeds in much the same way as a certain type of chemical reaction mentioned later. If x oz. be the weight for an average infant at age t mo., then

$$\frac{dx}{dt}=kx(a-x), \tag{21}$$

where k and a are constant for any racial stock.

Separating variables and using (3), p. 492, we find

$$\frac{dx}{x(a-x)}=k\,dt, \tag{22}$$

$$\log\frac{x}{a-x}=k_1(t-t_1), \tag{23}$$

where $k_1=ka$, and t_1 is the value of t at which $x=\frac{1}{2}a$. Or, if (23) be solved for x:

$$x=\frac{ae^{k_1(t-t_1)}}{1+e^{k_1(t-t_1)}}. \tag{24}$$

The adjacent table shows the observed average weight for a large group of British boys born in South Australia, — as compared with the theoretical weight calculated from a formula like (24).*

t	Obs.	Cal.
0	127	127
1	155	153
2	187	180
3	206	206
4	224	231
5	254	253
6	270	272
7	287	287
8	300	301
9	311	310

These formulas fit only the period of infancy, which, in man, is one of three "cycles of growth." The weight (x kg.) at any age (t yr.) is the sum, $x'+x''+x'''$, of partial weights contributed by the three cycles. Each partial weight is given by a formula of the same type, except that, after the infantile cycle, x' reaches some virtually constant value. [Cf. Ex. 7, 8, p. 50; Ex. 13, 14, p. 45.]

§ 196. Mortality Laws. The mortality tables used by insurance actuaries show the expected number of survivors at any age x, out of a large initial group.

Let l_x denote the number in the theoretical group living at age x. Then the instantaneous death rate is $-(dl_x/dx)$;

*See T. B. Robertson: *The Chemical Basis of Growth and Senescence;* also his *Principles of Biochemistry.*

and the percentage rate is $-(dl_x/dx) \div l_x$. This latter, called the "force of mortality," is denoted by μ_x:

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}. \tag{25}$$

By making reasonable assumptions as to how μ_x varies with x, formulas for l_x have been derived by integration, which closely fit experience.

The first important formula was derived by B. Gompertz in 1825. He assumed that, due to a gradual weakening of physical resistance, *μ_x increases with the age x in a geometrical progression*, and has the form

$$\mu_x = Bc^x, \tag{26}$$

where B and c are constants. From this he deduced the formula

$$l_x = kg^{c^x}, \tag{27}$$

where k and g are also constants. (See Ex. I below.)

To take account of accidental deaths without previous physical deterioration, M. W. Makeham later assumed that μ_x should be the sum of two terms: the value in (26) and a constant A. Thus

$$\mu_x = A + Bc^x. \tag{28}$$

From this he obtained

$$l_x = k\,s^x g^{c^x}, \tag{29}$$

where s also is constant. This formula, Makeham's Law, fits observed facts reasonably well, and is much used in smoothing or graduating tables.

Ex. I. Derive Gompertz's law (27) from (26).

Equating μ_x in (25) and (26), and separating variables:

$$\frac{1}{l_x}\,dl_x = -Bc^x\,dx.$$

$$\therefore \qquad \log l_x = -\frac{B}{\log c}\,c^x + \log k. \tag{30}$$

Then, since $\log l_x - \log k = \log(l_x/k)$, we have

$$\frac{l_x}{k} = e^{-\frac{B}{\log c}c^x} = \left(e^{-\frac{B}{\log c}}\right)^{c^x}. \tag{31}$$

Finally, calling the quantity in parenthesis g:

$$l_x = kg^{c^x}.$$

EXERCISES

1. Solve each following *D.E.* [Constant factors are better kept in a *numerator* with dx or dy, rather than thrown into a denominator.]

(*a*) $y\,dx - x\,dy = 0$, (*b*) $y^2\,dx - x^2\,dy = 0$,
(*c*) $3\,y\,dx - 2\,x\,dy = 0$, (*d*) $xy\,dx + 4\,dy = 0$.

Find the value of each arbitrary constant, if $y = 10$ when $x = 2$.

2. Find the relation between the variables named, in each case:

(*a*) Q and t, if $dQ = kQ\,dt$, and $Q = 100$ when $t = 0$;
(*b*) T and t, if $dT = -a(T-50)\,dt$, and $T = 150$ at $t = 0$;
(*c*) p and v, if $p\,dv + v\,dp = 0$, and $p = 15$ when $v = 200$;
(*d*) p and h, if $dp = -k\,p\,dh$, and $p = 30$ when $h = 0$;
(*e*) i and t, if $di = k(b-i)dt$, and $i = 0$ when $t = 0$.

3. The probability P that a molecule of gas will travel x cm. without a collision is given by $l\,dP = -P\,dx$, where l is constant. Find the formula for P, knowing that $P = 1$ when $x = 0$.

4. Du Noüy's *D.E.* for the decreasing surface tension (y) of a blood serum is: $2\sqrt{t}\,dy + ky\,dt = 0$. Solve, if $y = y_0$ at $t = 0$.

5. For two groups of British infants, the constants in (21) had the values below. Derive formulas (23) and (24) from (21) for each.

(*a*) Boys: $k = .000\,919$, $a = 318$; also, $x = 159$ at $t = 1.46$.
(*b*) Girls: $k = .000\,782$, $a = 312$; also, $x = 156$ at $t = 1.54$.

6. For a group of British boys and men, the weight (x kg.) at age t yr. was $x = 9 + x'' + x'''$, where x'' and x''' are determined by

$$\frac{dx''}{dt} = .00783\,x''(24 - x''), \qquad \frac{dx'''}{dt} = .00777\,x'''\,(35 - x'''),$$

with $x'' = 12$ at $t = 5.5$, and $x''' = 17.5$ at $t = 16$. Solve for x.

7. In (24) show that a is the limit of x as $t \to \infty$. Also verify that $x = \frac{1}{2}\,a$ when $t = t_1$.

8. Of what function is μ_x the derivative, by (25)?

9. Find a formula for l_x if the force of mortality runs thus:

(*a*) $\mu_x = h$, constant; (*b*) $\mu_x = h/(120 - x)$.

10. Derive Makeham's Law (29) from (28).

11. Express as an integral the total number of years lived by the theoretical group, between the ages x_1 and x_2, on the basis of (29).

12. Find the general solution of each following *D.E.*:

(*a*) $(1+u)dv+(1-v)du=0$, (*b*) $(y^2-9)dx+x\,dy=0$,
(*c*) $(x+y)dx+(x-y)dy=0$, (*d*) $(1+p^2)dq-(1+q^2)dp=0$,
(*e*) $\tan 2\,\theta\, d\rho=2\,\rho\, d\theta$, (*f*) $\sqrt{1+y^2}\,dx+\sqrt{1+x^2}\,dy=0$,
(*g*) $\sqrt{1-y^2}\,dx=3\,x^2y\,dy$, (*h*) $(4\,x-y)dx=(2\,x+y)dy$,
(*i*) $udz-zdu=\sqrt{z^2-u^2}\,du$, (*j*) $y\sec^2\theta\,d\theta=2\log y\,dy$,
(*k*) $4\,e^x\sin y\,dx=(e^x-1)\cos y\,dy$, (*l*) $(5\,xy^2-3\,x^3)dx=(x^2y+y^3)dy$,

(*m*) $\left(4\,x^2+y^2\cos\dfrac{y}{x}\right)dx=xy\cos\dfrac{y}{x}\,dy$,

(*n*) $\left(e^{\frac{2y}{x}}+\dfrac{y}{x}e^{\frac{y}{x}}\right)dx=e^{\frac{y}{x}}dy.$

13. Show that (11), p. 335, can be transformed into

$$x\sqrt{1-y^2}+y\sqrt{1-x^2}=\sin c.$$

[*Hint:* Let $\sin^{-1}x=\theta$, $\sin^{-1}y=\phi$, and expand $\sin(\theta+\phi)$.]

§ 197. **Linear and Extended Linear Equations.** A differential equation is called *linear* if the dependent variable y and dy/dx occur only linearly. The general form is

$$\frac{dy}{dx}+Py=Q, \tag{32}$$

where P and Q may be functions of x alone, or constants.

An equation which is like (32), except that the entire right or left side is multiplied by some power of y, is called an "extended linear equation." It is always reducible to a linear *D.E.*; but this is unnecessary.

Linear and extended linear equations can be solved in the following way: First form a simpler "preliminary" equation, like (32) but with its right member zero. Solve this by separating variables. Then modify the solution, so as to fit the more complicated given equation. This can be done by *changing the constant of integration to a variable*, properly chosen.*

* This idea of "variation of constants," originated by J. L. Lagrange, a Frenchman, is useful also in much more complicated problems.

Ex. I. $$\frac{dy}{dx} - y \operatorname{ctn} x = y^3 \csc x. \tag{33}$$

The preliminary equation is $dy/dx - y \operatorname{ctn} x = 0$, or

$$\frac{dy}{y} - \operatorname{ctn} x \, dx = 0. \tag{34}$$

The solution of (34) is $\log y - \log \sin x = \log C$, or

$$y = C \sin x.$$

Now replace C by a variable v; and try to determine v in such a way that

$$y = v \sin x \tag{35}$$

will be a solution of the *given* equation (33).

Substituting (35) in (33), we find that we must have

$$\left(v \cos x + \sin x \frac{dv}{dx}\right) - \operatorname{ctn} x (v \sin x) = \csc x (v \sin x)^3.$$

Or, since $v \cos x$ cancels with $-\operatorname{ctn} x(v \sin x)$:

$$\frac{dv}{v^3} = \sin x \, dx.$$

$$\therefore \qquad \frac{1}{v^2} = 2 \cos x + k. \tag{36}$$

The value of v from (36) makes (35) a solution of (33); viz.

$$y = \pm \frac{\sin x}{\sqrt{2 \cos x + k}}. \tag{37}$$

EXERCISES

Always notice first whether the variables are separable.

1. Find the general solution of each following *D.E.*:

(*a*) $dy/dx + 2y = e^{5x}$,
(*b*) $dy/dx - 4y = \sin 3x$,
(*c*) $dy/dx + 3y/x = e^{x^2}$,
(*d*) $dy/dx - 2y/x = x^2 + 5$,
(*e*) $x\, dy/dx + 2y = x^5 y^2$,
(*f*) $dy/dx - 3y = xy^3$,
(*g*) $y\, dy/dx + y^2/x = \cos x$,
(*h*) $dy/dx + 2xy = 3x$,
(*i*) $y^2\, dy/dx + y^3 \operatorname{ctn} x = \csc x$,
(*j*) $dy/dx + y \sec x = 5 \cos^2 x$,
(*k*) $(x^2+1) dy/dx - 2xy = y^2 \tan x$,
(*l*) $dy/dx + y \sin x = \sin 2x$,
(*m*) $x^3\, dy/dx + 4y = 12$,
(*n*) $x^2\, dy/dx + (x^2 + y^2) = 0$.

2. Find the solution of $dy + y \tan x\, dx = \sin 2x\, dx$, for which $y = 20$ at $x = 0$.

3. The speed (v cm./sec.) of a moving particle increased in this way: $dv/dt + 5v = 980$. Find v at any time, if $v = 0$ at $t = 0$.

4. A surface wound, of area S, heals thus: $dS/dt + kS(1 + t/p) = 0$, where k and p are constants depending upon the age and condition of the patient. Find the formula for S at any time, if $S = S_0$ at $t = 0$.

5. An electric current i varied thus: $di/dt + 200\,i = 800 \sin 400\,t$. Find i at any time, if $i = 2$ at $t = 0$.

6. Like Ex. 5 if $di/dt + 200\,i = 400$, and $i = 0$ at $t = 0$. Show that the graph of i as a function of t must be a curve like that in Fig. 112.

7. Show that the general solution of (32) is always

$$y = e^{-\int P\,dx}\left[c + \int Q\,e^{\int P\,dx}\,dx\right]. \tag{38}$$

8. Using (38) as a *formula*, find again the solutions of Ex. 1 (a), (b).

§ 198. Motion in a Resisting Medium. The force necessary to give a free object, of weight W, any acceleration a is: $F = (W/g)a$. For motion in a straight line: $a = dv/dt$, where v is the speed. Hence

$$F = \frac{W}{g}\frac{dv}{dt}. \tag{39}$$

But motion through air or any other medium encounters resistance. For large objects at ordinary speeds, this varies about as v^2. For minute particles, as in a spray, the resistance equals v times some constant R. The force effective in producing acceleration is, then, the difference between the *applied force* F and the resistance Rv. Thus $F - Rv$ replaces F in (39). Or we may write

$$\frac{W}{g}\frac{dv}{dt} + Rv = F. \tag{40}$$

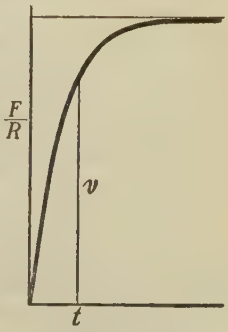


FIG. 112.

If we know how the applied force F varies with t, the solution of (40) will give

the speed at any time. If F is constant and the object starts from rest, v will increase toward a limiting value, viz. F/R as in Fig. 112.

When the resistance varies as the square of the speed v, the second term in (40) is changed to Rv^2; and the *D.E.* (40) ceases to be linear. However, the variables are separable if F is constant.

Rotation. The torque necessary to give a free object M any angular acceleration (α rad./sec^2.) about any axis OA is: $T = I\alpha$, where I is the moment of inertia of M with respect to OA. Or, calling ω the angular speed, $T = I\, d\omega/dt$. But when there is a frictional resistance whose torque is proportional to ω, say $R\omega$, we have

$$I\frac{d\omega}{dt} + R\omega = T. \tag{41}$$

If the applied torque T is a known function of t, the solution of (41) gives the angular speed at any time.

Ex. I. For a flywheel turning according to (41), find the manner of slowing down after the power is cut off.

Here $T = 0$, and we have a case of the Compound Interest Law:

$$\omega = ke^{-\frac{Rt}{I}},$$

where k is the value of ω at $t = 0$, — *i.e.*, at the cut-off.

§ 199. Electric Current with Self-Induction. By Ohm's law, the e.m.f. or electromotive force E required to drive an electric current i through a resistance R is

$$E = Ri. \tag{42}$$

But if i varies, an opposition e.m.f. arises by self-induction. This is proportional to the rate at which i changes; viz. it is $L\, di/dt$, where the constant L is the "coefficient of self-induction." The e.m.f. actually effective is the difference

between the "impressed" e.m.f. E and the induced e.m.f. $L\,di/dt$. Thus

$$E - L\frac{di}{dt} = Ri.$$

$$\therefore \qquad L\frac{di}{dt} + Ri = E. \tag{43}$$

If the impressed e.m.f. E varies with t in some known way, the solution of (43) will show how i varies.

Ex. I. Let E vary periodically: $E = E_0 \sin \omega t$.

Substituting this in (43), and solving as in Ex. 5, p. 341, or Ex. 12, p. 347, we obtain finally

$$i = ce^{-\frac{R}{L}t} + \frac{E_0}{R^2 + \omega^2 L^2}[R \sin \omega t - \omega L \cos \omega t], \tag{44}$$

where c is arbitrary unless initial values are given.

The form of (44) can be modified by introducing an auxiliary angle ϕ, whose cosine and sine are proportional to R and ωL respectively:

$$R = k \cos \phi, \quad \omega L = k \sin \phi. \tag{45}$$

Squaring and adding, we find k^2; and thus obtain

$$R = \sqrt{R^2 + \omega^2 L^2} \cos \phi, \quad \omega L = \sqrt{R^2 + \omega^2 L^2} \sin \phi. \tag{46}$$

Then in (44) the quantity in brackets takes the form

$$\sqrt{R^2 + \omega^2 L^2}\,[\sin \omega t \cos \phi - \cos \omega t \sin \phi],$$

and this new bracketed quantity is simply $\sin(\omega t - \phi)$.

$$\therefore \qquad i = ce^{-\frac{R}{L}t} + \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \phi). \tag{47}$$

Evidently the exponential term approaches zero in time, — rapidly if L is small in comparison with R. Thus i presently varies virtually as indicated by the second term, that is, periodically with the same period as the impressed e.m.f., but lagging behind the latter, as indicated by the auxiliary angle ϕ. This "lag" depends upon ω, L, and R; for, by (45), $\tan \phi = \omega L/R$.

With a condenser present, as in radio apparatus, a term q/C is added to the left member of (43), — where $q = \int i\,dt$. In terms of q, the *D.E.* takes the form in Ex. 4, p. 357.

EXERCISES

1. Find the solution of (40) for which $v=0$ at $t=0$, if $W=2.5$, $g=980$, $R=.01$, and $F=2.5$ constantly.

2. The same if F varies thus: $F=5 \sin t$.

3. The same if $F=10\,e^{-t}$.

4. Find the solution of (40) for which $v=100$ at $t=0$, if $W=2$, $g=980$, $R=.005$, and $F=0$ constantly.

5. Find the solution of (41) if $\omega=0$ at $t=0$, and if

(*a*) $I=200$, $R=500$, $T=1000$; (*b*) $I=1000$, $R=100$, $T=50$.

6. Derive the general solution of $5\,dy/dt+24\,y=50 \sin 2\,t$. Find the constant if $y=2.2$ when $t=0$. Express the solution in the form (47), and find the lag.

7. From either form of solution in Ex. 6, calculate y when $t=1$, 2.5, and 5. For still larger values of t, about what maximum and minimum values will y have?

8. Derive the solution of (43) for which $i=4$ at $t=0$, if $L=.02$, $R=3$, and $E=10 \sin 200\,t$. Get both forms, (44) and (47).

9. As an airplane runs along the ground to a stop, its speed v decreases thus: $M\,dv/dt=-F-kv^2$, where M, F, and k are constants. If $v=v_0$ at $t=0$, find v after t sec.

10. The radical stress p in the walls of a hollow cylinder varies thus with the distance r from the axis: $dp/dr+2\,p/r=k/r$, where k is constant. Solve this *D.E.*

§ 200. Chemical Dynamics. Some chemical changes proceed slowly enough for their progress to be measured and studied mathematically. The rate of change often varies in a simple manner, giving rise to a *D.E.* The decomposition of radium, and other cases coming under the *C.I.L.*, have been previously mentioned. We have space for only a few more illustrations.* The following definitions will help to clarify the equations.

* See Mellor, *Chemical Statics and Dynamics*; also Hitchcock and Robinson, *Differential Equations of Chemistry*.

(I) A *mol* is a weight, different for different substances, proportional to the weight of a molecule of any substance in question; viz. W gm. if the "molecular weight" is W.

The molecular weight of common salt is 58.5; and a mol of salt is 58.5 grams. The molecular weight of oxygen is 32; so a mol of oxygen is only 32 gm. But there are just as many molecules of oxygen in its mol of 32 gm., as there are molecules of salt in its mol of 58.5 gm. Likewise there are as many molecules in x mols of one as in x mols of the other.

(II) When a substance is dissolved, the number of mols in each liter is called the *concentration*.

(III) The "Law of Mass Action" states that the rate of combination of two reacting substances at any time is proportional to *the product of the concentrations of the reacting substances.*

§ **201. Bimolecular Reactions.** Let us suppose that two given substances, G and G', dissolved together, combine to form a product P (and possibly other products), in such a way that one molecule of G with one of G' forms one of P. What quantity of P will have been formed at any time, if the concentrations of G and G' at the start were a and a'?

Think of a single liter of solution. When enough molecules have combined to form x mols of P, then G and G' will each have lost x mols. There will remain $(a-x)$ mols of G, and $(a'-x)$ mols of G'. The rate of formation of P in mols is dx/dt. By (III), § 200, this is proportional to the product of $(a-x)$ and $(a'-x)$:

$$\frac{dx}{dt}=k(a-x)(a'-x). \tag{48}$$

The variables in this *D.E.* are separable. Integrating by (3), p. 492, if $a\neq a'$, gives

$$\frac{1}{a-a'}\log\frac{a-x}{a'-x}=kt+C. \tag{49}$$

Since $x=0$ at $t=0$, C is easily found. We obtain finally:

$$\frac{1}{a-a'}\log\frac{a'(a-x)}{a(a'-x)}=kt. \tag{50}$$

By dividing through by t, and substituting observed values for t and x, a chemist can test whether (50) fits a particular reaction. If it does, he will get virtually the same value for k, from each substitution.

§ 202. Side by Side Reactions. Consider now a substance G which decomposes simultaneously into two substances, P and Q, but at different rates, relatively fixed. Some molecules of G produce a like number of P; and others of G produce molecules of Q as numerous as themselves.

If a is the original concentration of G, and x and y are the present concentrations of P and Q, then the present concentration of G is $(a-x-y)$. Thus the rates of formation of P and Q are

$$\frac{dx}{dt}=k_1(a-x-y), \tag{51}$$

$$\frac{dy}{dt}=k_2(a-x-y). \tag{52}$$

By solving these *D.E.*'s, we find the amounts of P and Q formed at any time, per liter.

The method is to divide, get $y=k_2(x/k_1)$, substitute this in (51) and separate variables. Then get y directly from x.

§ 203. Consecutive Reactions. Suppose now that a substance G produces P; and P in turn produces Q at a different rate, molecule for molecule. Here the rate of formation of Q at any time is proportional to the concentration of P remaining at the moment.

Let a be the original concentration of G; and let x and y be the number of mols of P and Q, respectively, which have been produced thus far. Then the present concentrations of G and P are respectively $(a-x)$ and $(x-y)$. Hence

$$dx/dt = k_1(a - x), \tag{53}$$

$$dy/dt = k_2(x - y)\,; \tag{54}$$

and, at $t = 0$, both x and y were zero.

To solve these *D.E.*'s, first find x as a function of t from (53) and substitute its value in (54). The latter then becomes a linear *D.E.* of the first order, solvable as in § 197.

EXERCISES

1. Work out the solution of (48), if $a = 3$, $a' = 2$, and $k = .3$. Check, comparing with (50).

2. In Ex. 1 find at what time x will be .5; 1.0; 1.5; 1.9.

3. From (50) obtain x as a function of t.

4. Solve (48) when $a = a' = .5$ and $k = .02$. Find x when $t = 100$.

5. Solve (51) and (52), if $a = 4$, $k_1 = .3$, and $k_2 = .2$. Find x and y when $t = 2$.

6. The same as Ex. 5, if $a = .75$, $k_1 = .05$, and $k_2 = .1$.

7. Like Ex. 5, for the *D.E.*'s (53) and (54).

8. Solve (53) and (54), if $a = 3$, $k_1 = .2$, and $k_2 = .5$. Find x and y when $t = 3$.

9. The rate of recombination of ions in a certain problem is $dn/dt = -kn^2$. Find n at any time t, if $n = N$ at $t = 0$.

10. An equation met in studying digestion is: $dx/dt = \frac{1}{2}k^2Fq/x$, where k, F, and q are constants. Solve, with $x = X$ at $t = 0$.

11. For an electric current: $L = .004$, $R = 4$, $E = 8 \sin 60\,\pi t$. Express i as a function of t, in the form (47).

12. Derive (44) as the general solution of (43).

13. The path of a ray of light through the air, in a vertical plane, has the *D.E.*: $C d\rho/d\phi = \rho\sqrt{\mu^2\rho^2 - C^2}$, where μ is the index of refraction, varying with ρ. Show that if μ were constant, the path would be straight.

14. For a certain ascending airplane, the upward speed at any height (y ft.) is $34(1 - y/h)$, where h is a constant called "the height of the ceiling." (a) Express the time required to rise from $y = 0$ to $y = Y$. (b) If $t = 500$ when $y = 10000$, and $t = 1000$ when $y = 15700$, find h.

15. Find a value of k, for which $y = ce^{kx}$ is a solution of $d^2y/dx^2 - 6\,dy/dx + 8\,y = 0$. Is there more than one? (Cf. Ex. 2, p. 332.)

16. Like Ex. 13 for $d^3y/dx^3 - 2\,d^2y/dx^2 - 13\,dy/dx - 10\,y = 0$.

LINEAR EQUATIONS OF ANY ORDER WITH CONSTANT COEFFICIENTS

§ 204. The Basic Method. We next consider differential equations of higher order, which are linear, with constant coefficients. A typical specimen is the equation

$$\frac{d^3y}{dx^3}-4\frac{d^2y}{dx^2}-11\frac{dy}{dx}+30\,y=0, \tag{55}$$

which will be used to illustrate several principles.

Any similar linear *D.E.* of the first order, — say, $dy/dx+Py=0$, — has a solution of the form $y=ce^{-Px}$. We therefore ask whether an exponential equation,

$$y=ce^{kx}, \tag{56}$$

can be a solution of our higher *D.E.* (55). Testing this by substitution, as in § 193 and Ex. 15, p. 347, the left member of (55) reduces to

$$ce^{kx}(k^3-4\,k^2-11\,k+30).$$

This will be zero, regardless of the value of c, if

$$k^3-4\,k^2-11\,k+30=0. \tag{57}$$

That is, if we use any one of the three values of k satisfying this equation, $y=ce^{kx}$ will be a solution of (55).

Solving (57) by synthetic division: $k=2, 5$, or -3. Thus we have three solutions of (55):

$$y=ce^{2x},\quad y=ce^{5x},\quad y=ce^{-3x}.$$

In each case the various terms obtained on substituting in (55) would cancel out. The same would be true if we used the sum of these terms, with various coefficients c_1, c_2, c_3; viz.

$$y=c_1e^{2x}+c_2e^{5x}+c_3e^{-3x}. \tag{58}$$

As (58) involves three arbitrary constants, it is the general solution of the given *D.E.*

Consider now the general linear $D.E.$, of any order n, with constant coefficients $P_1, P_2, \cdots, P_n$:

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+P_{n-1}\frac{dy}{dx}+P_ny=0. \qquad (59)$$

The foregoing example shows that this differential equation will have $y=ce^{kx}$ as a solution, if k is any of the n roots of the auxiliary algebraic equation

$$k^n+P_1k^{n-1}+\cdots+P_{n-1}k+P_n=0, \qquad (60)$$

in which a power of k replaces a derivative of like order in (59). If $k_1, k_2 \cdots, k_n$ are the n roots of (60), then (59) has the solution

$$y=c_1e^{k_1x}+c_2e^{k_2x}\cdots c_ne^{k_nx}. \qquad (61)$$

If all these powers are *distinct*, (61) involves n arbitrary constants and is the general solution of (59). In certain cases this form (61) will be modified later.

In solving equations like (59) hereafter, we need not go through all the steps used for (55) above. Simply write the auxiliary equation (60) directly from (59), solve for k, and write the final solution (61).

§ 205. Equal Roots. If the auxiliary equation (60) has some of its n roots equal, (61) involves, in effect, fewer than n arbitrary constants. Thus, if k_1 is a repeated root, r-fold:

$$c_1e^{k_1x}+c_2e^{k_1x}\cdots+c_re^{k_1x} \qquad (62)$$

is really only *one* term: $(c_1+c_2+\cdots+c_r)e^{k_1x}$, or ce^{k_1x}.

Clearly the single constant c can assume all values obtainable by giving any values to the constants $c_1, \cdots, c_r$ individually.

It turns out that the general solution of (59) in such a case is obtained by replacing the repeating terms (62) by successive powers of x, times the common power of e; viz.

$$e^{k_1x}(c_1+c_2x+\cdots+c_rx^{r-1}). \qquad (63)$$

Including the ordinary term $c_1e^{k_1x}$, there are as many terms in this group as the order of the repeated root k.

No general proof of this principle will be given here. Ex. 11 below will throw some light on the matter. See also Ex. 23, p. 329, in which two exponents become equal.

Ex. I. $$\frac{d^4y}{dx^4}-24\frac{d^2y}{dx^2}-64\frac{dy}{dx}-48\,y=0. \tag{64}$$

The auxiliary equation is $k^4-24\,k^2-64\,k-48=0$, with roots -2, -2, -2, 6. Thus the general solution of (64) is

$$y=e^{-2x}(c_1+c_2x+c_3x^2)+c_4e^{6x}.$$

(How could this be checked?)

EXERCISES

1. Find the general solution of each following *D.E.*:

(*a*) $\dfrac{d^2y}{dx^2}-4\,y=0,$

(*b*) $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-2\,y=0,$

(*c*) $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6\,y=0,$

(*d*) $\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}-21\dfrac{dy}{dx}+45\,y=0,$

(*e*) $\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+4\,y=0,$

(*f*) $\dfrac{d^4y}{dx^4}-33\dfrac{d^2y}{dx^2}-100\dfrac{dy}{dx}-84\,y=0,$

(*g*) $\dfrac{d^2y}{dt^2}-10\dfrac{dy}{dt}+25\,y=0,$

(*h*) $\dfrac{d^3y}{dx^3}-9\dfrac{dy}{dx}+10\,y=0,$

(*i*) $2\dfrac{d^2y}{dt^2}+3\dfrac{dy}{dt}-y=0,$

(*j*) $3\dfrac{d^4y}{dx^4}+8\dfrac{d^3y}{dx^3}+6\dfrac{d^2y}{dx^2}-y=0,$

(*k*) $\dfrac{d^2y}{dt^2}-6\dfrac{dy}{dt}=0,$

(*l*) $\dfrac{d^3y}{dt^3}+8\dfrac{d^2y}{dt^2}=0.$

2. (*a*), (*b*). In Ex. 1 (*a*), (*b*), verify each solution, as in § 193.

3. Find the solution of $d^2y/dt^2-3\,dy/dt-4\,y=0$, which reduces to $y=7$ and $dy/dt=3$ at $t=0$.

4. A point moves vertically in such a way that $d^2y/dt^2+8\,dy/dt+7\,y=0$, where y is the height. Find y in terms of t, if $y=12$ and $dy/dt=0$ at $t=0$.

5. Find the solution of $d^2y/dx^2-9\,y=0$, for which $y=A$ and $dy/dx=3\,B$ at $x=0$. Reduce the solution to the form $y=A\cosh 3\,x+B\sinh 3\,x$. Verify directly that this form is a solution, and that it gives the specified values at $x=0$.

6. Like Ex. 5 for $d^2y/dx^2-16\,y=0$, if $y=M$ and $dy/dx=N$ at $x=0$.

7. Write the general solution of $d^2y/dx^2-4\,y=0$ immediately in terms of hyperbolic functions. Verify.

8. Like Ex. 7 for $d^2x/dt^2 - 25\,x = 0$.

9. For a direct electric current with leakage, the strength I varies thus with the distance x from the receiving end: $d^2I/dx^2 = a^2I$, where a is constant. Write the solution of this in hyperbolic functions, if $I = A$ and $dI/dx = B$ at $x = 0$.

10. Like Ex. 9 for the voltage E, with the same form of differential equation, and with $E = C$ and $dE/dx = D$ at $x = 0$.

11. Verify that $y = ce^{2x}$ is a solution of $d^3y/dx^3 - 6\,d^2y/dx^2 + 12\,dy/dx - 8\,y = 0$. Using Lagrange's method of variation of constants (§ 197), replace c by v, and find the most general value of v for which $y = ve^{2x}$ will be a solution of the given *D.E.* [Cf. (63).]

§ **206. Imaginary Roots.** When the auxiliary equation (60) has a pair of imaginary roots, say

$$k = a \pm bi, \qquad (i = \sqrt{-1}), \tag{65}$$

the corresponding exponential terms in (61) have the imaginary form

$$c_1e^{(a+bi)x} + c_2e^{(a-bi)x}. \tag{66}$$

It usually happens, however, that c_1 and c_2 have such values as to make the sum of these two terms real.

By proceeding as in § 155 and Ex. 7, 8, p. 267, we can change such terms to a partly trigonometric form, viz.

$$e^{ax}(C_1 \sin bx + C_2 \cos bx). \tag{67}$$

In actual practice we need not bother to write imaginary terms like (66), but may instead write immediately the corresponding trigonometric form (67) of the solution. If the solution ought to be real in fact as well as in form, the values of C_1 and C_2 will turn out real when determined.

Ex. I. Find the solution of the *D.E.*

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 425\,y = 0, \tag{68}$$

for which $y = 10$ and $dy/dx = 90$ at $x = 0$.

The auxiliary equation is $k^2 + 10\,k + 425 = 0$, with the roots $k = -5 \pm 20\,i$. Hence the general solution of (68) is

$$y = e^{-5x}(C_1 \sin 20\,x + C_2 \cos 20\,x). \tag{69}$$

At $x=0$ this gives $y=C_2$, $dy/dx=20\,C_1-5\,C_2$. To fit the values specified above,

$$C_2=10, \qquad 20\,C_1-5\,C_2=90.$$

Hence $C_1=7$, $C_2=10$; and the required solution is

$$y=e^{-5x}(7\sin 20\,x+10\cos 20\,x). \tag{70}$$

Remarks. (I) By the method of § 199, Ex. I, this can be put also into the form:

$$y=\sqrt{149}\,e^{-5x}\sin(20\,x+\phi), \tag{71}$$

where $\phi=\tan^{-1}(10/7)$.

(II) Terms like e^{-5x} in a solution are non-periodic. If a linear *D.E.* with constant coefficients is to have a periodic solution, k must have pure imaginary values. These will produce pure sine and cosine terms.

§ 207. Damped Vibrations. Often a particle P, moving along a straight line, is pulled toward a fixed point O on the line, by a force proportional to the distance OP; but at the same time encounters a resistance proportional to its speed v. Denoting the distance OP (or "displacement") by y, positive in one direction, we may write $v=dy/dt$, $dv/dt=d^2y/dt^2$. Hence equation (40), § 198, becomes:

$$\frac{W}{g}\frac{d^2y}{dt^2}+R\frac{dy}{dt}=-k^2y.$$

(As to the $-$ sign see the remarks about Fig. 109, § 192.)

Transposing, and changing the notation slightly:

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0. \tag{72}$$

The values of k in the auxiliary equation are

$$k=\frac{-b\pm\sqrt{b^2-4\,ac}}{2\,a}. \tag{73}$$

Oscillations are possible only if $b^2<4\,ac$, — that is, for relatively small resistances. For large values of b, P merely *approaches* O, and gradually slows down.

Ex. I. A particle moved according to this *D.E.*:

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+10\,y=0. \tag{74}$$

Its displacement and speed at the start were respectively $y=6$ and $dy/dt=0$. Study the motion.

The solution, obtained by the method of § 206, is

$$y=e^{-t}\,(2\sin 3\,t+6\cos 3\,t), \tag{75}$$

or

$$y=\sqrt{40}\,e^{-t}\sin\,(3\,t+\phi), \tag{76}$$

where $\phi=\tan^{-1}3$. Also, from (75):

$$\frac{dy}{dt}=-20\,e^{-t}\sin 3\,t. \tag{77}$$

Successive maxima and minima occur when $\sin 3\,t=0$, or $t=0,\ \pi/3,\ 2\,\pi/3,\ \pi$, etc. At those times, by (75),

$$y=e^{-t}(\pm 6),\ =6,\ -6\,e^{-\frac{\pi}{3}},\ 6\,e^{-\frac{2\pi}{3}},\ -6\,e^{-\pi},\ \cdots.$$

The third maximum of y after $t=0$ is only $6\,e^{-2\pi}$; or .015, approx. The "damping factor" e^{-t} rapidly reduces the "amplitude." Note also that y becomes zero at regular intervals of $\pi/3$ sec., beginning at $-\phi/3+\pi/3$. Figure 113 shows how y varies with t.

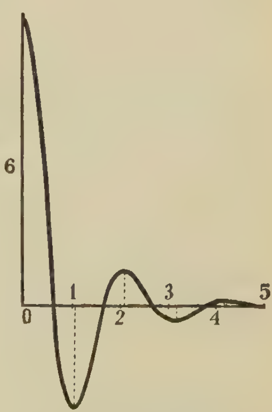


Fig. 113.

EXERCISES

1. Solve each following *D.E.* (Here y', y'', etc., denote dy/dx, d^2y/dx^2, etc.)

(*a*) $y''+4\,y=0$,

(*b*) $y'''-2\,y'-4\,y=0$,

(*c*) $y''+6\,y'+10\,y=0$,

(*d*) $y^{\text{iv}}-10\,y'''+y''-100\,y=0$,

(*e*) $y'''-4\,y'+15\,y=0$.

2. Solve $d^4y/dx^4+8\,d^2y/dx^2+16\,y=0$. [In the form (63) for repeated roots, we may replace e^{kx} terms by trigonometric forms.]

3. The same as Ex. 2 for $y^{\text{v}}-2\,y^{\text{iv}}+2\,y'''-4\,y''+y'-2\,y=0$.

4. A certain pendulum swings approximately thus: $d^2\theta/dt^2 + .2\, d\theta/dt + 64.01\, \theta = 0$. If $\theta = .08$ and $d\theta/dt = 0$ at $t = 0$, derive the formulas corresponding to (75) and (77) above. Find the value of θ at its eighth maximum, after $t = 0$. Draw a rough graph by inspection.

5. Like Ex. 4, for the vibrating end of a spring, the *D.E.* being $4\, d^2y/dt^2 + 4\, dy/dt + 101\, y = 0$, with $y = .2$ and $dy/dt = 0$ at $t = 0$.

6. Find the solution of each of the following, if $y = 2$ and $dy/dt = 8$ at $t = 0$:

(*a*) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 13\, y = 0,$ (*b*) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3\, y = 0,$

(*c*) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4\, y = 0,$ (*d*) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} = 0,$

7. (*a*), (*b*). Plot y as a function of t in Ex. 6 (*a*), (*b*), taking $t = 0$, .5, 1.0, $\cdots$, 3.0; and also finding where y has any maximum, minimum, or zero value.

8. Under certain conditions a galvanometer swings thus: $d^2\theta/dt^2 + 2\, k d\theta/dt + n^2\theta = 0$. Will it swing through the zero point if $k > n$? Write the general solution if $k < n$.

9. With a condenser in the circuit, an electric current varies thus: $L\, d^2i/dt^2 + R\, di/dt + i/C = de/dt$, where L, R, and C are constants, and e is the e.m.f. If $e = E$, a constant, and $R^2 - 4\, L/C = -4\, L^2\omega^2$, write a formula for i at any time t.

§ 208. Treatment of a Right Member. A linear *D.E.* often involves a term free from y and its derivatives, — written as the right member of the equation. One way to handle such cases is shown in the following example.

Ex. I. $$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20\, y = \sin t. \tag{78}$$

(*A*) First find what the solution would be if the right member were zero:

$$y'' + 4\, y' + 20\, y = 0. \tag{79}$$

For this preliminary *D.E.*, the auxiliary equation is

$$k^2 + 4\, k + 20 = 0. \tag{80}$$

This gives $k = -2 \pm 4\, i$; and the solution of (79) is

$$y = e^{-2t}(C_1 \sin 4\, t + C_2 \cos 4\, t). \tag{81}$$

This last does not satisfy (78), for these terms would cancel out if substituted in the left member, and would leave nothing to equal the right member. But if we can find any quantity at all, which, upon substitution in the left member of (78) would give merely $\sin t$, — and thus be a *particular* solution, — the terms in (81) can be added to it, and the combination still be a solution. Moreover, that combination will be the *general* solution, since it will involve the two arbitrary constants C_1 and C_2. The terms in (81) are called the "Complementary Function."

(B) Next differentiate (78) twice, getting

$$\frac{d^4y}{dt^4}+4\frac{d^3y}{dt^3}+20\frac{d^2y}{dt^2}=-\sin t. \tag{82}$$

Combining (82) and (78) to eliminate the right member:

$$\frac{d^4y}{dt^4}+4\frac{d^3y}{dt^3}+21\frac{d^2y}{dt^2}+4\frac{dy}{dt}+20\,y=0. \tag{83}$$

Any value of y which satisfies (78) must satisfy (82) also, and hence be some solution of (83).

The auxiliary equation for (83) is $k^4+4\,k^3+21\,k^2+4\,k+20=0$. And since we know by (80) that one factor of this is $k^2+4\,k+20$, we easily find the other factor by division; viz. $k^2+1=0$. Thus the roots are: $k=\pm i$, besides the previous values $-2\pm 4\,i$.

The general solution of (83) is

$$y=e^{-2t}(C_1\sin 4\,t+C_2\cos 4\,t)+C_3\sin t+C_4\cos t. \tag{84}$$

And the particular solution of (78) which we seek is included among these terms. It is not in the first two which, as we have seen, form the Complementary Function. It is, therefore, one or both of the remaining terms.

(C) Hence substitute $y=C_3\sin t+C_4\cos t$ back in the original *D.E.* (78), and see for what special values of C_3 and C_4 this latter value of y will be a solution. [Cf. Ex. 3(a) p. 332.] The substitution gives

$$(19\,C_3-4\,C_4)\sin t+(4\,C_3+19\,C_4)\cos t.$$

These terms reduce to the right member ($\sin t$) only if

$$19\,C_3 - 4\,C_4 = 1, \quad 4\,C_3 + 19\,C_4 = 0. \tag{85}$$

Solving these equations simultaneously gives

$$C_3 = \tfrac{19}{377}, \quad C_4 = -\tfrac{4}{377}.$$

Hence, finally, the general solution of the given *D.E.* (78), including the complementary function, is

$$y = e^{-2t}(C_1 \sin 4\,t + C_2 \cos 4\,t) + \tfrac{19}{377} \sin t - \tfrac{4}{377} \cos t. \tag{86}$$

If the original right member were something else than $\sin t$, we should proceed somewhat differently. But our general plan would be the same: to eliminate the right member, either by differentiation alone, or by combining with the original *D.E.* some equation or equations obtained through differentiation.

§ 209. **Impressed Oscillations.** Suppose that the acceleration of the vibrating free end P of a spring, if the other end were fixed, would be

$$\frac{d^2y}{dt^2} = -4\,\frac{dy}{dt} - 20\,y. \tag{87}$$

And suppose that the support to which the other end is attached oscillates along the same line, with an acceleration

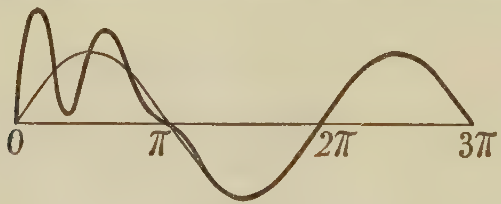


Fig. 114.

equal to $\sin t$. The acceleration of P in space will be the sum of $\sin t$ and the value in (87).

$$\therefore \qquad \frac{d^2y}{dt^2} + 4\,\frac{dy}{dt} + 20\,y = \sin t. \tag{88}$$

This is the same as the *D.E.* (78), solved in § 208. Hence the general solution (86) describes the motion here. Figure 114 shows graphically the character of the oscillations, for certain initial values which determine C_1 and C_2. The natural vibrations of the spring, — given by the terms in (86) which involve $\sin 4t$ and $\cos 4t$, and which have the period $\pi/2$, — virtually die out in time, leaving only the oscillations of period 2π due to the support.

EXERCISES

1. Find the general solution of each following *D.E.* (Here the accents denote derivatives with respect to x.)

(*a*) $y''-4y=5\sin x$, (*b*) $y''+6y'+9y=\cos 2x$,
(*c*) $y''+9y=\sin 5x$, (*d*) $y''-7y'+6y=e^{5x}$,
(*e*) $y''+16y=\cos 4x$, (*f*) $y^{\text{iv}}-5y''+4y=7e^{-2x}$,
(*g*) $y''+25y=6x-8$, (*h*) $y''-4y'+4y=x^2$,
(*i*) $y^{\text{iv}}+b^4y=ax$, (*j*) $y''+a^2y=x\,e^x$.

2. Find the solution of $d^2y/dt^2-5\,dy/dt+4y=6e^{2t}$, for which $y=2$ and $dy/dt=20$ at $t=0$.

3. Solve $d^2y/dt^2+4y=8\sin 2t$, if $y=2$ and $dy/dt=0$ at $t=0$.

4. The electric charge in a radio condenser varied thus: $Ld^2q/dt^2+Rdq/dt+q/C=E$. Take $L=2\times10^{-5}$, $R=.8$, $C=10^{-9}$, $E=20{,}000\sin\omega t$, $\omega=7\times10^6$; denote by bi the imaginary part of k; and express q at any time, if $q=0$ and $dq/dt=0$ at $t=0$.

5. The upper end of a vertical spring is fixed, and an external force F is applied to the lower end P. The internal restoring force is proportional to the elongation y; the resistance to motion is proportional to the speed. Show that the *D.E.* for the motion of P has the form

$$\frac{d^2y}{dt^2}+a\frac{dy}{dt}+by=cF. \tag{89}$$

6. Find the general solution of (89) if $a=10$, $b=425$, and $cF=10\sin 5t$. How does the natural period of vibration under the internal forces compare with the period of the external force? Which period will dominate, in time?

7. Like Ex. 6, but with $cF=10\sin 20t$. What will happen to the spring, in time?

8. Like Ex. 7 but with $a=0$ and $b=400$.

EQUATIONS OF SPECIAL TYPES

§ 210. Exact Equations. If a *D.E.* has the form

$$M\,dx+N\,dy=0, \tag{90}$$

where M and N are functions of x and y, it is readily solved in case

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}. \tag{91}$$

We need merely find a function $f(x, y)$ having the left member of (90) as its exact differential, and then write

$$f(x, y)=c. \tag{92}$$

In § 49 we saw that if $M\,dx+N\,dy$ is to be an exact differential, then (91) must hold. The converse is readily proved, also; but we can work without it. We may simply (1) find what the integral function $f(x, y)$ must be, *if any exists;* and (2) test whether the function so found will, upon differentiation, actually give $M\,dx+N\,dy$.

Ex. I. $$(5\,x^4+6\,x^2y^2)dx+(4\,x^3y-7\,y^6)dy=0. \tag{93}$$

If there be a function $f(x, y)$ having this as its exact differential, then $\partial f/\partial x=5\,x^4+6\,x^2y^2$. Hence f must contain the terms

$$x^5+2\,x^3y^2. \tag{94}$$

Also, necessarily, $\partial f/\partial y=4\,x^3y-7\,y^6$. Hence f must involve

$$2\,x^3y^2-y^7. \tag{95}$$

We therefore test as a possible solution of (93):

$$x^5+2\,x^3y^2-y^7=c. \tag{96}$$

And evidently this is correct. [Differentiate (96) and see.]

Ex. II. $$(5\,x^4-6\,x^2y^2)dx+(4\,x^3y-8\,y^4)dy=0. \tag{97}$$

Here $\partial M/\partial y=-12\,x^2y$, while $\partial N/\partial x=12\,x^2y$. The foregoing process cannot work. [But (97) is homogeneous, of degree 4; hence the variables could be separated by putting $y=vx$.]

§ 211. The type $y''=f(y)$. If a *D.E.* has the form

$$\frac{d^2y}{dx^2}=f(y), \tag{98}$$

it can be solved by the following method.

Denote dy/dx by l. Then $d^2y/dx^2 = dl/dx$. But

$$\frac{dl}{dx} = \frac{dl}{dy} \cdot \frac{dy}{dx} = \frac{dl}{dy} l. \tag{99}$$

Replace d^2y/dx^2 by $l\,dl/dy$; and (98) will give

$$l\,dl = f(y)dy. \tag{100}$$

Let $F(y)$ denote any integral of $f(y)dy$. Then

$$l^2 - 2\,F(y) = C. \tag{101}$$

Taking the square root, and replacing l by dy/dx, we can separate variables and get the final solution.

§ 212. **Loaded Cords.** If a cord or wire supports a distributed load in equilibrium, the tensions T and T', which act at the ends of any tiny length ds, must balance one another and the load dW on ds. The horizontal components of T and T' must then be equal. Call each H. (Fig. 115.) The difference of the vertical components, $V' - V$, must equal dW. But $V = Hl$, and $V' - V = H\,dl$, where l is the slope. Hence

$$H\,dl = dW. \tag{102}$$

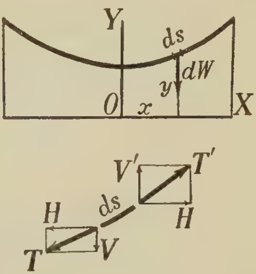


FIG. 115.

If the loading per unit length, including the weight of the cord as stretched, is a constant w, then $dW = w\,ds$. And by (66), § 81, $ds = \sqrt{1+l^2}\,dx$. Hence (102) becomes

$$H\,dl = w\sqrt{1+l^2}\,dx. \tag{103}$$

It is easy to separate variables, integrate, and express l as a function of x. Then, since $l = dy/dx$, another integration gives the equation of the curve $y = f(x)$. [See Ex. 8–9, p. 360.]

The curve in this case is called the *catenary*.

§ 213. **Remarks on Chapter VIII.** Given any *D.E.*, first note its order and degree. If of the first order, can it be solved for dy/dx? Is it separable, exact, or homogeneous? Is it linear with any coefficients, or of the extended linear

type? If of higher order, is it linear with constant coefficients? Or of the type $y'' = f(y)$?

If none of these, it may be solvable only by approximate methods, *e.g.*, assuming the Maclaurin series for y, — say $y = A + Bx + Cx^2 + \cdots$, — and determining the constants $A, B, C, \cdots$, so as to satisfy the *D.E.* This plan is discussed in texts on differential equations.

Our further applications of calculus will lie chiefly in two fields, geometry and kinematics (the science of motion). We first take up some further principles of Analytic Geometry.

EXERCISES

1. Solve each following *D.E.*:

(*a*) $(8x^3 + 3x^2y^3)dx + (3x^3y^2 - 6y^5)dy = 0$, with $y = 2$ when $x = 1$;

(*b*) $(2x \log y + ye^x)dx + (x^2/y + e^x)dy = 0$, with $y = 1$ when $x = 0$;

(*c*) $(2x - 3y)dx + (3x + 2y)dy = 0$, with $y = 0$ when $x = 1$.

2. Find the solution of $d^2y/dx^2 + 4y = 0$, for which $y = 10$ and $dy/dx = 0$ at $x = 0$: (*a*) by the method of § 211; (*b*) by another method. Compare.

3. Solve $d^2y/dx^2 = 9/\sqrt{y}$, if $y = 4$ and $dy/dx = 0$ at $x = 0$.

4. The same as Ex. 3, for $d^2y/dx^2 = 8/y^2$.

5. If a particle P, outside the atmosphere, falls straight toward the center of the earth C, its acceleration at any distance y mi. from C is: $d^2y/dt^2 = -k/y^2$, where $k = 95{,}040$, approx. If $y = 200{,}000$ and $dy/dt = 0$ at $t = 0$, find t for any other value of y. When will $y = 5000$?

6. When an object falls rapidly through the air, its speed v varies thus: $dv/dt - Rv^2 = -k/y^2$, where y is the distance from the center of the earth, and R and k are constants. By the method of § 211, express v at any y [v is considered negative].

7. The plate of Ex. 8–9, p. 200, will swing thus about OA: $d^2B/dt^2 = -a \sin B$, where $a = 1024\sqrt{2}\, g/3150$. Express t in terms of B as an integral. [The latter could be found only by approximation.]

8. Separate the variables in (103), integrate, and derive the equation of the catenary, if $y = k$ and $dy/dx = 0$ at $x = 0$. Effect the integrations by means of hyperbolic functions.

9. Solve Ex. 8 by using logarithmic and exponential forms.

10. In Fig. 115 let the loading per *horizontal* unit be constant. Show that the curve is then a parabola.

EXERCISES FOR REVIEW

1. As thin ice forms, the thickness x varies thus: $dx/dt = k/x$. Show that x is proportional to the square root of the time elapsed.

2. As a large mass of air rises rapidly, its temperature T falls thus: $dT/dh = -kT/(a-h)$. Express T at any elevation h.

3. In a certain water pipe the velocity v parallel to the axis, at any point, varies thus with the distance r from the axis:

$$\frac{d}{dr}\left[r\frac{dv}{dr}\right] = -kr.$$

Express v as a function of r, if $v = 0$ at $r = a$, and v is finite at $r = 0$.

4. The *D.E.* for the Brownian movement of a particle due to molecular bombardment is reducible to the form

$$\frac{m}{2}\frac{dz}{dt} - \frac{RT}{N} = -\frac{Kz}{2},$$

where m, R, T, N, K are constants, and z is a variable determining the displacement. Find z as a function of t.

5. The value of an investment grows at a varying percentage rate r. Write a formula for V at any time, if $V = P$ at $t = 0$.

6. A chemical reaction runs thus: $dx/dt = .08(3-x)$, $dy/dt = .2(x-y)$. Find y as a function of t, if $x = y = 0$ at $t = 0$.

7. In the steady flow of heat in a long thin rod, the temperature T varies thus: $d^2T/dx^2 = b^2T/h^2$, where b and h are constants. Express T at any distance x, if $T = 100$ at $x = 0$, and $T = 50$ at $x = 10\,h/b$.

8. A "simple pendulum" consists of a heavy particle P, swinging at the end of an inextensible weightless cord without resistance. Show from Fig. 116 that the *D.E.* for the motion of P is $l\,d^2\theta/dt^2 = -g\sin\theta$.

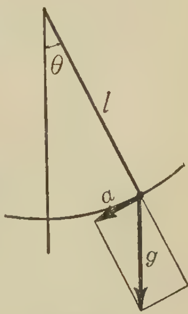


FIG. 116.

9. In Ex. 8 find the angular speed at any time. Also express t in terms of θ at any point. [The integral is reducible to a standard elliptic integral of the first kind. § 180.]

10. Find the general solution in each case:

(a) $\dfrac{dy}{dx} - 2xy = x^3y^2$, (b) $x^2\dfrac{dy}{dx} + (2xy - y^2) = 0$,

(c) $\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} + 5\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 4y = 60\sin 2x$.

CHAPTER IX

FURTHER METHODS IN PLANE ANALYTIC GEOMETRY

PART I. ALGEBRAIC PROCESSES

§ 214. Distance from a Line to a Point. We shall often need to know how far some point (X, Y) is from a line

$$Ax+By+C=0. \tag{1}$$

If the line is not horizontal, the term Ax will be present; and A can always be made positive. Then, as proved below, if we substitute (X, Y) in the left member of (1) and divide by $\sqrt{A^2+B^2}$, *the result will be numerically equal to the required distance.* If the result is $+$, (X, Y) lies to the right of the line; if $-$, to the left.

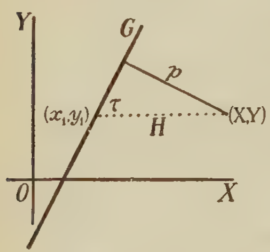


FIG. 117.

E.g., the distance of (3, 2) from the line $7x+11y-9=0$ is

$$p=\frac{7(3)+11(2)-9}{\sqrt{7^2+11^2}}=\frac{34}{\sqrt{170}},$$

and (3, 2) lies to the right of the line. (Check by plotting.)

Proof of the Method. From (X, Y) draw a horizontal line H meeting the given line G at (x_1, y_1). (Fig. 117.) If (X, Y) lies to the right, $X>x_1$, and the length of the line H is $X-x_1$. Hence the required perpendicular distance is:

$$p=(X-x_1)\sin\tau. \tag{2}$$

By inspection of (1) the slope of G is

$$\tan\tau = -\frac{A}{B}, \quad \text{whence } \sin\tau = \frac{A}{\sqrt{A^2+B^2}},$$

$$p = \frac{(X-x_1)\,A}{\sqrt{A^2+B^2}} = \frac{AX - Ax_1}{\sqrt{A^2+B^2}}. \tag{3}$$

But as (x_1, Y) lies on the line G, we have by (1):

$$-Ax_1 = BY + C.$$

Substituting in (3) gives finally, as stated above:

$$p = \frac{AX+BY+C}{\sqrt{A^2+B^2}}. \tag{4}$$

If τ is obtuse, it is replaced in (2) by its supplementary acute angle, but the sine is the same, and (4) still holds.

If (X, Y) lies to the left of the given line, $(X-x_1)$ is the negative of the actual horizontal distance in Fig. 117, — which makes the final result in (4) negative.

If the given line G is horizontal, $A=0$. We may then write the equation of G in the form $y \pm k = 0$. Substituting Y will give the distance p numerically; and a positive result will show that (X, Y) is *above* G.

EXERCISES

1. How far is each following point from the specified line, and in which direction? Check roughly by plotting.

(*a*) (8, 4), from $5x+12y+29=0$; (*b*) (7, 5), from $2x-19=0$;
(*c*) (5, −2), from $4x-3y-12=0$; (*d*) (7, 5), from $y+4=0$;
(*e*) (3, 5), from $y=x+10$; (*f*) (2, − 1), from $y-4=0$;
(*g*) (0, 0), from the line joining (1, 9) and (− 7, 3);
(*h*) (0, − 6), from the line through (1, 1) with slope −1;
(*i*) (− 2, − 3), from the line through (0, 0) perpendicular to $x=3y$.

2. Find the equation of a circle with center (8, − 2):
(*a*) if tangent to $5x-12y=25$; (*b*) if tangent to $7x+y=0$.

3. A force of 10 lb. acts along the line $4x+3y=3$. Find its torque about an axis perpendicular to the XY-plane at (7, 5).

4. The same as Ex. 3, for the line $x+2y=8$ and the point (0, 0).

5. Referred to certain axes, with 1 mi. as the unit distance, a water main follows the line $4x-3y+6=0$. How far away is a house, located at (4.5, 7)?

6. A rafter R at one end of an attic follows the line $2x+3y=26$. A window is to have one corner at (4, 5). How far is this from R?

7. Express by an equation the fact that a point (h, k) is equidistant from the lines $3x-4y+2=0$ and $5x+12y=0$, being located to the right of each.

8. Find the center (h, k) of the circle inscribed in the triangle whose sides are: $3x-4y+9=0$, $5x+12y-13=0$, $4x+3y-28=0$. [Hint: Equating one distance expression to each of two others, with properly chosen signs, show that: $h-8k=-13$, $7h-k=19$. Then solve for h and k.]

9. Find the equation of the circle in Ex. 8.

10. Find the moment of inertia, with respect to the line $4x-3y=20$, of a thin, flat plate, of surface density k, bounded by $y=x^2$ and $y=4$. [Hint: A particle $k\,dy\,dx$ located at (x, y) is how far from the given line?]

11. Find the surface area generated by revolving about the line $2x-y=20$ the arc of $y=x^2$ between (0, 0) and (2, 4). [Show that an element of the surface can be expressed as

$$\frac{2\pi}{\sqrt{5}}(20-2x+x^2)\sqrt{1+4x^2}\,dx.]$$

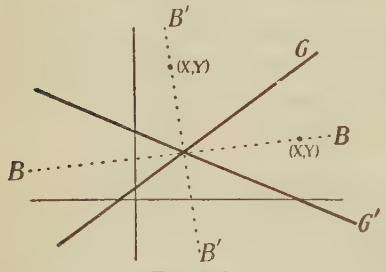


FIG. 118.

§ 215. Bisector of an Angle. To find the equation of the bisector of an angle between two straight lines, we may use the fact that any point (X, Y) on a bisector is equidistant from the two lines.

Ex. I. Suppose that the lines (G and G', in Fig. 118) are:

$$3x-4y+1=0, \quad 5x+12y-17=0. \tag{5}$$

The distances of (X, Y) from these lines are given (aside from possible − signs) by

$$p_1=\frac{3X-4Y+1}{5}, \quad p_2=\frac{5X+12Y-17}{13}. \tag{6}$$

The bisector B of one pair of angles between G and G' lies everywhere either to the right of both G and G' or else to the left of both. The expressions for p_1 and p_2 in (6) therefore have like signs all along B. Along the other bisector B', the signs are opposite; to make them agree when equating, we change the sign of p_1 or p_2.

At all points on the first bisector B, then,

$$\frac{3X-4Y+1}{5}=\frac{5X+12Y-17}{13}; \tag{7}$$

and along B',

$$\frac{3X-4Y+1}{5}=-\frac{5X+12Y-17}{13}. \tag{8}$$

For points on neither bisector, neither equation holds.

Simplifying (7) and (8), and writing (x, y) instead of (X, Y) for any point on either bisector, we have as the required equations:

$$x-8y+7=0, \quad 8x+y-9=0. \tag{9}$$

Remark. The slopes of these bisectors are $\frac{1}{8}$ and -8, respectively. What does this show?

§ **216. Angle between Lines.** It is useful to know a formula for the size of an angle between two lines.

To be definite, we define "the angle which a first line makes with a second" as the angle from (II) around to (I), taken counterclockwise, less than π.

By *Intro.*, § 284, this angle $K_{1,2}$ is given by

$$\tan K_{1,2}=\frac{l_1-l_2}{1+l_1l_2}, \tag{10}$$

where l_1 and l_2 are the slopes of lines (I) and (II).

By (10) we can also find the angle at which one *curve* crosses another; *i.e.*, the angle between the tangents.

Ex. I. Find the angle which the parabola $y=x^2$ makes with the line $y=2x$ at their higher intersection.

Solving simultaneously gives two intersections: (0, 0), (2, 4). We use (2, 4). The slope of the parabola is $l_1 = dy/dx = 2x = 4$; that of the line is $l_2 = 2$.

$$\tan K_{1,2} = \frac{4-2}{1+(4)(2)} = \frac{2}{9}; \quad K_{1,2} = 12° \, 32'.$$

§ 217. Point of Division. It is useful to have a formula also for a point $D(x', y')$ which divides a line segment P_1P_2 in some ratio $m_1 : m_2$. (Fig. 119.)

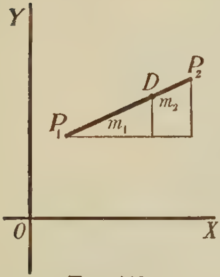


FIG. 119.

By similar triangles we easily get $(x'-x_1)/m_1 = (x_2-x')/m_2$. This and the corresponding equation for the y's give, on reducing:

$$x' = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \quad y' = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}. \quad (11)$$

Either of two given points may be regarded as P_1; but m_1 then refers to the adjacent segment P_1D.

A point D on the extension of P_1P_2 in either direction is said to divide P_1P_2 in some negative ratio. The first segment P_1D (*from the beginning of the line to the point of division*) runs in one direction; the second segment DP_2 (*from the point of division to the end of the line*) runs in the opposite direction. If one of the segments is considered negative, and either m_1 or m_2 is given a negative value in (11), these formulas will work correctly.

EXERCISES

1. Find the equations of the bisectors of the angles between $4x+3y-5=0$ and $5x-12y+15=0$. Show them perpendicular.

2. The same as Ex. 1 for $2x+y-5=0$ and $x-2y-10=0$.

3. Find the angle which a line of slope $\frac{4}{3}$ makes with one of slope $\frac{1}{2}$. Check by measurement.

4. Find the angle which $3x-4y=12$ makes with $2x+3y=0$.

5. By using (10) verify that either line in (9) actually does bisect one angle between the lines in (5).

6. Find the bisector of one angle between $x+3y=8$ and $3x-y+2=0$. Check by (10).

7. Find the angle which the first of each following pair of lines or curves makes with the second, at the highest intersection:

(*a*) $y=x^2$, $y=x+6$; (*b*) $y=x^3$, $y=5x-2$;
(*c*) $y=3x$, $x-2y=1$; (*d*) $y=x^3$, $y=x^2$;
(*e*) $y=4x$, $y^2=8x$; (*f*) $y=2x$, $x^2+y^2=10x$.

8. A point (x, y) moves in such a way that its distance from $3x-4y=0$ is always three times its distance from the X-axis. Find the equation of its path. What sort of locus?

9. A point $P(x, y)$ moves so that the lines joining it to $A(2, 0)$ and $B(-2, 0)$ intersect always at an angle of 45°. Find the equation of its path. Interpret. [Hint: After expressing the slopes of AP and BP, impose the necessary relation by (10).]

10. The same as Ex. 9, if the lines AP and BP are perpendicular.

11. Find a point (x', y') which divides the line from $(1, 4)$ to $(13, 10)$ in each of the ratios, $1:3$, $2:1$, $5:-4$, $-5:4$, $-4:5$, $-2:3$. Plot in each case and see that the results are reasonable.

12. A triangle has the vertices $(3, -6)$, $(9, -3)$, $(6, 12)$. Find the point which lies two-thirds of the way from each vertex to the midpoint of the opposite side.

13. If $m_1=m_2$ in (11), what values have x' and y'? Check.

14. A thin horizontal plate, of surface density k, is bounded by the X-axis and the upper half of $x^2+y^2=4$. Find the torque of its weight, about the line $x+y=10$. [After expressing the distance R, use polar coördinates.]

§ 218. Locus Problems. (Rectangular Coördinates.) Any problem concerning the locus of a point in a given plane can be studied analytically as follows:

(1) Choose axes in such a way that important points have simple coördinates, and important lines simple equations, — as far as possible. Denote by (x, y) or (X, Y) the point whose locus is required; and by a, b, c, k, or x_1, y_1, etc., any constants, or coördinates of fixed points.

(2) Express any given geometric condition by using a standard formula, — *e.g.*, the formula for distance, slope, angle between lines, etc., as the case may be. Simplify the equation algebraically as much as possible.

(3) Interpret the final equation and determine its full

geometric significance. Often a simple geometric proof of the correctness of the result is easy after we have discovered analytically what the locus should be.

Sometimes only part of a curve, whose equation has been obtained as that of the locus, will fit the problem as stated; but a slight modification or *generalization* of the statement may make the entire curve admissible.

We can often be guided in these matters by making some *experimental constructions* roughly, before starting the analytic solution outlined above. Simply draw the variable figure lightly, in several positions; and mark the moving point in question distinctly each time.

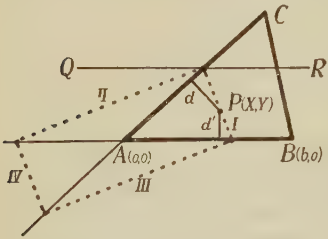


Fig. 120.

Ex. I. Find the locus of a point P whose distances from two sides of a given triangle (or those sides extended) have a constant sum k.

Experiment. Using sides AB and AC (Fig. 120), we mark several positions of P for which $d+d'=k$. The locus appears to be four sides of a certain rectangle, with vertices located on lines AB and AC, each at a distance k from the other line. As the problem does not specify the particular pair of sides AB and AC, the complete locus involves three such apparent rectangles.

Analytic Solution. Take line AB as the X-axis and A as $(0, 0)$, so that AC will be a line through the origin.

The equation of AC may then be written

$$ax+by=0 \quad \text{(with } a \text{ positive)}.$$

By (4), § 214, the distance of P from AC is

$$d=\pm\frac{aX+bY}{\sqrt{a^2+b^2}}. \tag{12}$$

(The $-$ sign applies when P lies to the left of AC.)

The distance of P from AB is Y, or $-Y$ if P lies below AB. Putting $d+d'=k$, as required, gives finally:

$$\pm\frac{aX+bY}{\sqrt{a^2+b^2}}\pm Y=k. \tag{13}$$

All four combinations of the $+$ and $-$ signs are admissible.

Interpretation. Each of the four equations (13) is linear. Hence the locus of (X, Y) is four straight lines; or, rather, limited parts thereof, since neither distance can exceed k.

If we regarded a "distance" as negative on one side of a line, — so that formula (4), § 214, always gave it correctly rather than its negative, — the other distance could then exceed k. A single unlimited line would be the locus, instead of four limited segments.

To recognize the four lines above, write (13) thus:

$$\pm\frac{aX+bY}{\sqrt{a^2+b^2}}=k-Y, \qquad \text{[two lines]}, \tag{14}$$

$$\pm\frac{aX+bY}{\sqrt{a^2+b^2}}=k+Y, \qquad \text{[two others]}. \tag{15}$$

The left member of each equation in (14) gives the distance of (X, Y) from AC, in some position; the right member, the distance from QR, — a line parallel to, and k units above, AB. These distances being equated, (14) represents the bisectors (I and II) of the angles between AC and QR. Likewise (15) represents the bisectors (III and IV) of the angles between AC and a line parallel to AB, k units below it.

These four bisectors must be mutually perpendicular, — as can be shown by elementary geometry or by their slopes.

Geometric Proof. Since any point P on segment I is equidistant from AC and QR, the sum of its distances from AC and AB must equal the constant distance k between AB and QR. Similarly for II, III, IV.

§ 219. **Loci in Polar Coördinates.** In some locus problems polar coördinates are best.

Ex. I. The legs of a right triangle of constant area vary in length but lie along two fixed lines OA and OB. Find the locus of the foot of the perpendicular drawn from the vertex of the right angle to the hypotenuse.

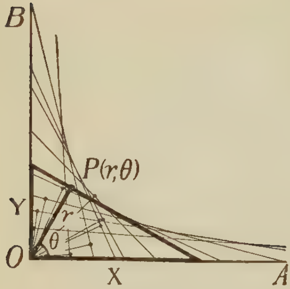


Fig. 121.

Experiment. Constructions give a loop, as in Fig. 121, — or four such loops if we allow the legs X and Y to run to the left and downward from O. The point O is evidently to be excluded from the loops.

Analysis. Choose the origin and notation as shown. Then

$$X = r \sec\theta, \qquad Y = r \csc\theta.$$

By hypothesis $\frac{1}{2}XY$ is some constant; call this k^2. Then $\frac{1}{2}(r\sec\theta)(r\csc\theta) = k^2$. Hence

$$r^2 = 2\,k^2 \sin\theta\cos\theta; \tag{16}$$

i.e.,

$$r^2 = k^2 \sin 2\,\theta. \tag{17}$$

Interpretation. This resembles $r^2 = 100\cos 2\,\theta$, plotted in Fig. 12, p. 21. In fact, since the sine of any angle is equal to the cosine of an angle 90° smaller, we may write (17) thus:

$$r^2 = k^2\cos(2\,\theta - 90°) = k^2\cos 2(\theta - 45°). \tag{18}$$

Hence the locus of Fig. 121 is one loop of the *Lemniscate* of Fig. 12, but turned 45° upward, tangent to OA and OB. In general, four such loops or two complete lemniscates (except for O), constitute the locus.

Remark. As we have given no geometric definition of the lemniscate, no separate geometric proof is available here.

§ 220. **Parametric Forms.** In some locus problems it is helpful to employ auxiliary variables or parameters. These

may later be eliminated, — or retained and the curve studied by parametric equations.

Ex. I. The base of a triangle is fixed, and the opposite angle constant, while the vertex moves. Find the locus of the center of the inscribed circle.

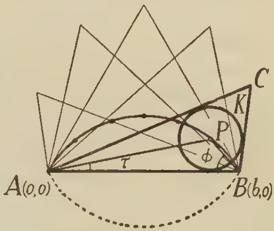


Fig. 122.

Experiment. Construction indicates a circular arc through A and B (Fig. 122); or two equal opposite arcs if the figure be reflected in line AB.

Analysis. The moving center P lies on the bisectors of the variable base angles. We select as parameters the halves of these angles, denoted by τ and ϕ and subject to the relation:

$$2\tau+2\phi+K=180°. \tag{19}$$

The equations of bisectors AP and BP are then

$$y=x\tan\tau, \qquad y=-(x-b)\tan\phi. \tag{20}$$

We may eliminate both τ and ϕ at once. By (19),

$$\tau+\phi=90°-\tfrac{1}{2}K.$$

Hence $$\tan(\tau+\phi)=\text{ctn}\,\tfrac{1}{2}K.$$

And, by the formula for $\tan(A+B)$, p. 490, this becomes

$$\frac{\tan\tau+\tan\phi}{1-\tan\tau\tan\phi}=\text{ctn}\,\tfrac{1}{2}K. \tag{21}$$

But from (20):

$$\tan\tau=\frac{y}{x}, \quad \tan\phi=-\frac{y}{x-b}.$$

Substituting these values in (21) and simplifying:

$$\frac{-by}{x^2+y^2-bx}=\frac{1}{\tan\frac{1}{2}K}. \tag{22}$$

Clearing of fractions and completing both squares:

$$\left(x-\frac{b}{2}\right)^2+\left(y+\frac{b}{2}\tan\tfrac{1}{2}K\right)^2=\frac{b^2}{4}\sec^2\tfrac{1}{2}K. \tag{23}$$

Interpretation. The locus is a circular arc with

$$\text{center}\ \left(\frac{b}{2}, -\frac{b}{2}\tan\tfrac{1}{2}K\right);\quad \text{radius}\ \frac{b}{2}\sec\tfrac{1}{2}K.$$

This center O lies on the perpendicular bisector of AB; also on a line through A with inclination $-\frac{1}{2}K$, that is, perpendicular to AC when $\triangle ACB$ is isosceles. (Fig. 123.) The radius equals OA, hence the circle passes through A and B.

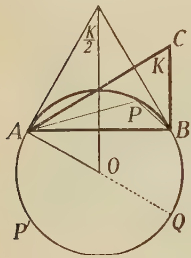


FIG. 123.

Geometric Proof. $\angle APB$, the supplement of $\tau+\phi$, equals $90°+\frac{1}{2}K$. Being constant, it is inscribed in some circular arc (APB). Also, it must be measured by half an arc ($AP'B$) equal to $180°+K$. And, since the diameter through A makes arc $AP'Q = 180°$, hence arc $QB=K$. Thus $\angle OAB$ must be $\frac{1}{2}K$, as stated.

Remark. Arc $AP'B$ is the locus of the center of an *escribed* circle, tangent to AB and to sides AC and BC extended. If the problem specified merely tangency to three lines, instead of the limited portions which are sides of a triangle, arc $AP'B$ would be part of the locus.

EXERCISES *

1. Find the locus of a point P if the feet of perpendiculars drawn from it to the X- and Y-axes are always 10 units apart.

2. Find the locus of the center of a circle which

(a) Passes through the points (4, 1) and (12, 5);

(b) Is tangent to the lines $15x - 8y = 0$ and $3x+4y=0$;

(c) Is tangent to the Y-axis and cuts off a segment of length 10 units on the X-axis.

3. Find the locus of a point P whose distances from $y=3x$ and $y=-3x$ have a constant product 10.

4. Find the locus of the vertex C of a triangle, whose base AB is fixed in length and position, if

(a) The sum of the squares of the sides AC and BC is constant;

* Suggestion: Keep some of these to work occasionally from day to day.

(*b*) The ratio of AC and BC is a constant, m/n;

(*c*) The slopes of AC and BC have a constant product k;

(*d*) The distance from A to the midpoint of BC is constant;

(*e*) $\angle B = 2\angle A$. [Use the formula for tan $2A$, p. 490.]

5. Find the locus of the midpoint of an ordinate of

(*a*) The parabola $y = x^2$, (*b*) The parabola $y^2 = 4px$.

6. Find the locus of a point P which lies one-third of the distance from a fixed point O to any point on a fixed circle:

(*a*) If O lies on the circle; (*b*) If O lies outside. [See (17), (18), p. 23.]

7. Find the locus of a point P between a fixed point O and any point Q of a fixed straight line:

(*a*) If P lies two-thirds of the distance from O to Q;

(*b*) If the product $\overline{OP} \cdot \overline{OQ}$ is a constant k^2.

8. Given the points $A(-4, 0)$, $B(-2, 0)$, $C(2, 0)$, $D(4, 0)$. Find the locus of a point P, if $\overline{PA} \cdot \overline{PD} = \overline{PB} \cdot \overline{PC}$ continually.

9. The same as Ex. 8, if $\overline{PA} \cdot \overline{PC} = \overline{PB} \cdot \overline{PD}$.

10. A straight line of length 20 moves with its ends A and B on the X- and Y-axes respectively. Find the locus of a point P which

(*a*) Is the midpoint of AB; (*b*) Divides AB in the ratio $2:3$;

(*c*) Is the foot of the perpendicular from the origin to AB;

(*d*) Is the third vertex of a right triangle ABP with legs $AP = 16$ and $BP = 12$.

11. Any line is drawn parallel to the base AB of a triangle, cutting the sides AC and BC in M and N respectively. Find the locus of

(*a*) The intersection of AN and BM;

(*b*) The midpoint of a rectangle with MN as one base and with the opposite base in the line AB.

12. Find the locus of P if $\angle APC = \angle CPD$, for the points in Ex. 8.

13. Find the locus of the intersection of perpendiculars from the ends of the fixed base AB of a triangle to the opposite sides:

(*a*) If the vertex C moves so as to keep $\angle ACB = k$, a constant;

(*b*) If C moves along a line parallel to AB and k units away.

§ 221. Rotation of Axes. To recognize some equations, we need to know the effect of rotating the X- and Y-axes through any angle ϕ.

Let P be any point, with coördinates (x, y) referred to the old axes, and (x', y') referred to the new. Then $x =$

$r \cos \theta = r \cos (\theta' + \phi)$. Expanded, this gives $x = r(\cos \theta' \cos \phi - \sin \theta' \sin \phi) = x' \cos \phi - y' \sin \phi$. Likewise y or $r \sin \theta$ reduces to $x' \sin \phi + y' \cos \phi$.

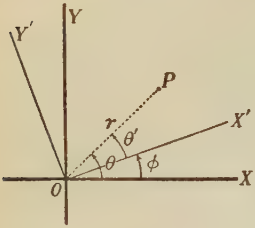


FIG. 124.

Thus the effect of the rotation is to replace every

$$\begin{aligned} \boldsymbol{x} &\text{ by } \boldsymbol{x} \cos \boldsymbol{\phi} - \boldsymbol{y} \sin \boldsymbol{\phi}, \\ \boldsymbol{y} &\text{ by } \boldsymbol{x} \sin \boldsymbol{\phi} + \boldsymbol{y} \cos \boldsymbol{\phi}. \end{aligned} \tag{24}$$

Special Case. An important special case is rotation through 45°. This replaces

$$x \text{ by } \frac{x-y}{\sqrt{2}}, \quad y \text{ by } \frac{x+y}{\sqrt{2}}. \tag{25}$$

Moreover, $(x+y)$, $(x-y)$, and xy are all replaced by simple expressions. Hence, when an unfamiliar equation involves $x+y$ or $x-y$ conspicuously, or involves xy without x^2 or y^2, it is well to try a 45° rotation.

Ex. I. $(x-y)^2 - 40(x+y) + 400 = 0.$ (26)

Rotating 45°, $x-y$ becomes $-2\,y/\sqrt{2}$ or $-y\sqrt{2}$; and $x+y$ becomes $x\sqrt{2}$. Hence the new equation is

$$(-y\sqrt{2})^2 - 40(x\sqrt{2}) + 400 = 0,$$

i.e., $\qquad y^2 = 20\sqrt{2}(x - 5\sqrt{2}).$

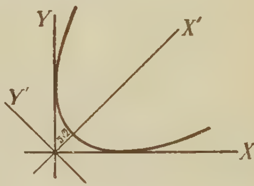


FIG. 125.

The curve is a parabola, extending along the new X-axis, and with its vertex at $(5\sqrt{2}, 0)$. Its directrix is the new Y-axis; for the distance from the vertex to the directrix must be one-fourth of the coefficient $20\sqrt{2}$.

It happens to be tangent to the original X- and Y-axes, as may be seen from the fact that $y=0$ gives $x=20$, 20 (two coincident intersections); and likewise $x=0$ gives $y=20$, 20.

Ex. II. Through what angle ϕ should the axes be rotated to transform $5\,x^2 + 24\,xy - 2\,y^2 = 100$ into an equation containing no xy term?

Replacing each x and y by its expression in (24):

$$5(x \cos \phi - y \sin \phi)^2 + 24(\cdots)(\cdots) - 2(\cdots)^2 = 100. \quad (27)$$

Multiplied out, the terms involving the new product xy are found to be

$$(-10 \cos \phi \sin \phi + 24 \cos^2\phi - 24 \sin^2\phi - 4 \sin \phi \cos \phi)xy.$$

We seek an angle ϕ for which this will reduce to zero:

i.e., $$-24 \sin^2 \phi - 14 \cos \phi \sin \phi + 24 \cos^2 \phi = 0. \quad (28)$$

Dividing by $-2 \cos^2 \phi$, and putting $\sin \phi / \cos \phi = \tan \phi$:

$$12 \tan^2 \phi + 7 \tan \phi - 12 = 0.$$

$$\therefore \qquad \tan \phi = \frac{-7 \pm \sqrt{49+576}}{24} = \frac{3}{4}, \ -\frac{4}{3}. \quad (29)$$

There is one possible acute angle, viz. 37°, approx.

From $\tan \phi = \frac{3}{4}$, we get at once $\sin \phi = \frac{3}{5}$, $\cos \phi = \frac{4}{5}$; and these could be used in (27) to get the simplified equation.

§ **222. Normal Equation of a Straight Line.** It is sometimes useful to have the equation of a line L in terms of the length p and direction angle A of its *normal*, — *i.e.*, of the perpendicular from the origin to the line.

If the X-axis lay along the normal, the equation of L would be $x = p$. Rotating the axes through a negative angle, $\phi = -A$, to their present position (Fig. 126) replaces x by $x \cos(-A) - y \sin(-A)$. Simplified, this gives

$$x \cos A + y \sin A = p. \quad (30)$$

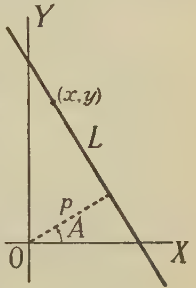


FIG. 126.

This is called the "normal form" of the equation of a straight line.

Any linear equation $ax + by + c = 0$ can be reduced to the form (30) by dividing by $\sqrt{a^2+b^2}$ with a suitably chosen sign.

Ex. I. Reduce $2x+3y+7=0$ to the normal form.

Transpose the 7 and divide by $-\sqrt{2^2+3^2}$:

$$-\frac{2}{\sqrt{13}}x-\frac{3}{\sqrt{13}}y=\frac{7}{\sqrt{13}}. \tag{31}$$

This gives a positive number in the right member. Also, since $-2/\sqrt{13}$ and $-3/\sqrt{13}$ are numbers whose squares have the sum 1, they are the cosine and sine of some angle A:

$$\cos A=-\frac{2}{\sqrt{13}}, \qquad \sin A=-\frac{3}{\sqrt{13}}.$$

Hence (31) has the required form, $x\cos A+y\sin A=p$

Here A lies in Quadrant III, approximately $236°\,19'$; and $p=7/\sqrt{13}$. The line is about opposite L, in Fig. 126, with respect to O.

Remark. If B be the angle which the normal makes with the Y-axis, (30) can be written also in the form

$$x\cos A+y\cos B=p. \tag{32}$$

EXERCISES

1. Transform by rotating the axes 45°:

(*a*) $4(x+y)^2+9(x-y)^2=36$, (*b*) $xy+40=0$,

(*c*) $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=20^{\frac{2}{3}}$, (*d*) $x^3+y^3=30\,xy$.

2. Transform $6x^2+12xy+y^2=2$ by rotating the axes through an acute angle, $\phi=\sin^{-1}(2/\sqrt{13})$. What sort of curve?

3. In equation (27) fill in the missing terms, multiply out, and verify (28).

4. Through what acute angle ϕ should the axes be rotated in each following case to obtain an equation containing no xy term?

(*a*) $8x^2-24xy+y^2+2x=0$, (*b*) $xy-50=0$,

(*c*) $3x^2-2xy+3y^2+10=0$, (*d*) $x^2+2xy+y^2=20$.

5. (*a*)–(*d*). In Ex. 4(*a*)–(*d*) carry out the rotation, and get the simplified equation.

6. Rotate the axes so as to simplify the equation $(x-y)^4+(x+y)^4=a^4$. Is the curve symmetrical in any way?

7. Show directly from equation (26) that its locus must be symmetrical with respect to the line $y = x$.

8. Rationalize the equation $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 20^{\frac{1}{2}}$. Compare the result with (26); also with Fig. 26 (D), p. 62.

9. A point (x, y) moves, but remains equidistant from (10, 10) and $x + y = 0$. Find the equation of its path. [Cf. (26).]

10. Derive (30) from the polar equation (15), p. 22.

11. Write the equation of a straight line:

(a) If its normal has the length 20 and the direction angle 45°;
(b) If it bisects Quadrants I and III;
(c) If its normal has the length 10 and the direction angle 300°.

12. Reduce to the normal form each following equation, and state how the line is situated:

(a) $4x+3y=10$, (b) $2x-5y-12=0$,
(c) $2x+7=0$, (d) $x+y=0$.

Why the ambiguity in (d), — geometrically speaking?

13. Show that the degree of an equation cannot be raised by a rotation of axes. From this show also that it cannot be lowered.

PART II. TANGENTS AND NORMALS

§ 223. The Differential Triangle. To apply the calculus to analytic geometry effectively, we need to know the strictly correct geometric meaning of dx, dy, and ds.

Let s be the length of a curve, from some fixed point A to any point $P(x, y)$. Let the tangent line PT at P be considered as running in the direction in which s increases; and let τ be its direction angle. (Fig. 127. Cf. § 2.) Then, whatever the direction of PT, $\tan\tau$ equals the slope, or derivative at P:

$$\tan\tau = \frac{dy}{dx}. \tag{33}$$

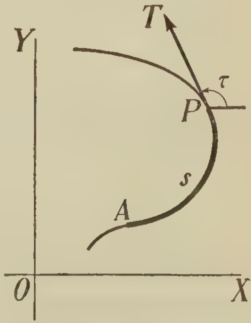
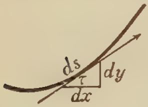


FIG. 127.

This relation is shown correctly by the differential triangle of Fig. 27, p. 63; or by either triangle of Fig. 128, with legs dx and dy, and hypotenuse along the tangent line.



Further, by *Intro.*, § 293, (9), we know that $(ds/dx)^2 = 1 + (dy/dx)^2$, or

$$ds^2 = dx^2 + dy^2. \qquad (34)$$

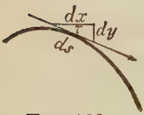


FIG. 128.

Hence each hypotenuse in Fig. 128 must be ds.

The differential triangle now shows also that

$$\sin \tau = \frac{dy}{ds}, \qquad \cos \tau = \frac{dx}{ds}. \qquad (35)$$

For our definition of τ, these relations are correct whether y and x increase or decrease with s. But if either decreases, the negative sign of the corresponding radical must be taken in (36) below.

Using in (35) values of ds/dy and ds/dx found from (34), and reducing, we have also

$$\sin \tau = \pm \frac{1}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}}, \qquad \cos \tau = \pm \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}. \qquad (36)$$

N.B. We ought not to depend upon memory for (33)–(36); but rather fix the *idea* of Fig. 128 clearly in mind. Then we can at any time draw a figure quickly, and merely read off desired relations.

For rapid use, we may regard a short arc of the curve as coinciding with ds on the tangent line, and not stop to draw the latter. Cf. Fig. 49, p. 135.

§ 224. Tangent Length and Subtangent. Important properties of certain curves relate to the length of the tangent line between the point of tangency and the X-axis, T_L in Fig. 129. Others relate to

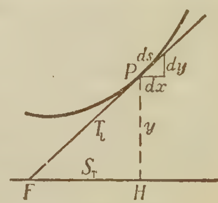


FIG. 129.

the "subtangent," S_T, which is the projection of the tangent length T_L upon the X-axis.

The differential triangle $[dx, dy, ds]$ is similar to the large triangle $[S_T, y, T_L]$. Hence

$$\frac{S_T}{y}=\frac{dx}{dy} \text{ and } \frac{T_L}{y}=\frac{ds}{dy}.$$

Multiplying through by y and reducing:

$$S_T=\frac{y}{\dfrac{dy}{dx}}; \qquad T_L=y\sqrt{1+\left(\frac{dx}{dy}\right)^2}. \tag{37}$$

For a falling curve S_T lies to the left of F and is then considered negative, agreeing with its value in (37).

It is best not to memorize (37) but obtain these values by similar triangles as above, when needed.

§ 225. **Normal Length and Subnormal.** The line perpendicular to the tangent at the point of tangency is called the normal to the curve. Its length to the X-axis is the "normal length," N_L, and the projection of the latter upon the X-axis is the "subnormal," S_N (Fig. 130).

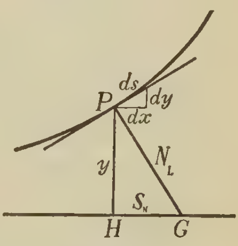


FIG. 130.

By comparing similar triangles, as in § 224, we find

$$S_N=y\frac{dy}{dx}, \qquad N_L=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}. \tag{38}$$

For a falling curve, S_N lies to the left of H, and is regarded as negative. The formula in (38) is still correct.

EXERCISES

1. From Fig. 128 read off ctn τ, sec τ, csc τ.

2. Derive (37) and (38) in detail. Do these formulas meet the homogeneity test?

3. For each following curve find S_T, T_L, S_N, N_L, at the point specified:

(*a*) $y=x^2$, at $(3, 9)$; (*b*) $y=x^3$, at $(2, 8)$;
(*c*) $y^2=12x$, at $(3, 6)$; (*d*) $y^2=x^3$, at $(1, 1)$;
(*e*) $y=\sin x$, at $\left(\frac{\pi}{6}, .5\right)$; (*f*) $y=\cosh x$, at $x=2$;
(*g*) $y=\log x$, at $(e, 1)$; (*h*) $y=2^x$, at $(1, 2)$.

4. In the parabola $y^2=4px$, at any point (x_1, y_1), show that
(*a*) $S_T=2x_1$; (*b*) $S_N=2p$.

5. Show that the reciprocal of the subtangent, at any point of the graph of a quantity Q, equals the instantaneous percentage rate at which Q is changing, per unit change in the independent variable. Show that this does not depend upon the scale used in plotting Q.

6. Some bacteria increased in number as in Table I. Plot, and measure the subtangent at $t=2$ and at $t=5$. What was the percentage rate of change of N at these times?

7. Like Ex. 6 for the angular speed ω of a flywheel, at $t=1$ and $t=3$. (Table II.)

8. The value of a property decreased after t yr. as in Table III. Find roughly the percentage rate of decrease per year, at $t=9$.

9. Find the equation of a curve if the subtangent or subnormal at every point has a value as follows:

Table I		Table II		Table III	
t	N	t	ω	t	V
0	100	0	10	0	30,000
1	165	10	6.1	2	28,000
2	272	20	3.7	4	22,000
3	448	30	2.2	6	12,000
4	739	40	1.4	8	5,200
5	1218	50	.8	10	2,400
6	2009	60	.5	12	1,200
7	3312	70	.3	14	800

(*a*) $S_T=k$, (*b*) $S_T=kx$, (*c*) $S_T=ky$,
(*d*) $S_N=k$, (*e*) $S_N=kx$, (*f*) $S_N=ky$.

§ 226. The Subtangent in Economics. In studying "demand curves" in Economics, the subtangent is useful.

If y be the demand for a commodity (*i.e.*, the amount that can be sold) at any price x, the "coefficient of elasticity of demand" (E) is defined as $(-dy/y)\div(dx/x)$:

$$E=-\frac{x}{y}\frac{dy}{dx}. \tag{39}$$

This is usually positive, y decreasing as x increases. (Fig. 131.) Note also that S_T is here negative.

If we denote the actual numerical value of S_T by S, (39) may be written: $E = x/S$. Thus, the coefficient E equals the ratio of the abscissa to the subtangent length, at any point on the demand curve.

Remark. Economists consider demand as a function of price, as here; but usually reverse the graph, plotting the price vertically and the demand horizontally, — a procedure of doubtful value.

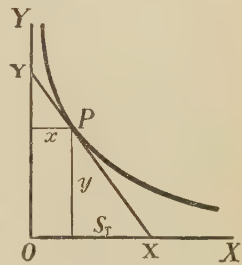


FIG. 131.

§ 227. Equations of Tangent and Normal. The slope of the tangent to a curve, at any point (x_1, y_1), is found by substituting x_1 and y_1 for x and y in the derivative. To emphasize this substitution, we write dy_1/dx_1 instead of dy/dx. The slope of the normal must then be $-dx_1/dy_1$. Knowing the slope and the point (x_1, y_1), we may write the equation of either line:

Tangent: $$y - y_1 = \frac{dy_1}{dx_1}(x - x_1), \tag{40}$$

Normal: $$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1). \tag{41}$$

Note that (x, y) is *any* point on the tangent or normal in question, while (x_1, y_1) is merely the point of tangency, on the curve. It is highly important to avoid confusing these. We should carefully substitute in the derivative before using (40) or (41).

Ex. I. Find the equations of the tangent and normal to the ellipse $400\,x^2 + 625\,y^2 = 250{,}000$ at (15, 16).

By (59), p. 75: $\dfrac{dy_1}{dx_1} = -\dfrac{800\,x_1}{1250\,y_1} = -\dfrac{800(15)}{1250(16)} = -\dfrac{3}{5}.$

Hence the required equations are

Tangent: $$y - 16 = -\tfrac{3}{5}(x - 15), \tag{42}$$

Normal: $$y - 16 = \tfrac{5}{3}(x - 15). \tag{43}$$

These reduce, respectively, to $3x + 5y = 125$ and $5x - 3y = 27$.

As a check observe that (15, 16) satisfies both equations.

§ 228. Intercept Properties. For various curves the intercepts cut off on the X- and Y-axes by any tangent or normal have special properties. These intercepts are easily found by letting $y=0$ or $x=0$ in equation (40) or (41). They can then be combined and studied as desired. To illustrate, consider the following

Theorem. A tangent to the hyperbola $xy=50$ forms with the axes a triangle of area 100, regardless of the point of tangency.

Proof. By (59), p. 75, the slope at any point (x_1, y_1) is $-y_1/x_1$. Hence the equation of the tangent is

$$y-y_1=-\frac{y_1}{x_1}(x-x_1). \tag{44}$$

When $x=0$, $$y=y_1-\frac{y_1}{x_1}(-x_1)=2\,y_1.$$

This is Y, the intercept on the Y-axis (Fig. 132). Similarly we find $X=2\,x_1$; and the area of the required triangle is

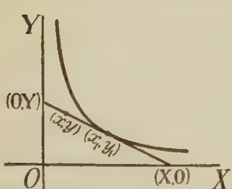


FIG. 132.

$$\tfrac{1}{2}\,XY=\tfrac{1}{2}(2\,x_1)(2\,y_1)=2\,x_1y_1. \tag{45}$$

But as (x_1, y_1) lies on the curve $xy=50$, we have $x_1y_1=50$.

$$\therefore \qquad \tfrac{1}{2}\,XY = 100. \qquad \text{Q. E. D.}$$

EXERCISES

1. Write the equations of the tangent and normal to each following curve at the point specified:

(*a*) $y=x^2$, at (2, 4);
(*b*) $x^2+y^2 = 100$, at (8, − 6);
(*c*) $y^2=x^3$, at (4, 8);
(*d*) $x^2-4\,y^2=81$, at (15, 6);
(*e*) $y=\sin x$, at (0, 0);
(*f*) $x^3+y^3=4\,xy$, at (2, 2);

(g) $y=\dfrac{x}{x^2+1}$ at (1, .5); (h) $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$, at $(-4, \frac{9}{5})$;

(i) $y^2=4\,px$, at (x_1, y_1); (j) $x^2+y^2+4\,y=0$, at (x_1, y_1).

2. How far is the point (3, -2) from the line which is tangent to $(x-4)^2+(y-3)^2=25$ at (7, -1)?

3. Like Ex. 2 for (5, 1) and the tangent to $x^{\frac{1}{2}}+y^{\frac{1}{2}}=6$ at (9, 9).

4. A triangle has one vertex at (9, 6); and its opposite side is the portion of the line tangent to $3\,y=16-x^2$ at (2, 4), which lies in the first quadrant. Find its area.

5. If we had neglected to substitute (x_1, y_1) in the derivative, in (44), what degree would the resulting erroneous equation of the tangent line have had?

6. Show that the intercepts of any tangent to the parabola $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$ have the constant sum a.

7. Show that, for any tangent to the hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, the portion included between the axes has the constant length a.

8. In Fig. 131 show that E is also equal to the ratio of the tangent lengths to the axes, viz. $YP \div PX$.

9. For the system of plotting mentioned in the Remark, § 226, show that $E = S/x$; also that $E = PX \div YP$.

10. If $E = n$, a constant, show that the equation of the "demand curve" is $x^n y = C$. [Use (39).]

§ 229. Direction in Polar Coördinates. When working with polar coördinates, we can best describe the direction of a curve at any point (r, θ) by telling what angle it makes with its radius vector. This angle is denoted by ψ (Greek letter "psi") and is found by the following formula, proved in § 230 below:

$$\tan\psi=\frac{r}{\dfrac{dr}{d\theta}}. \qquad (46)$$

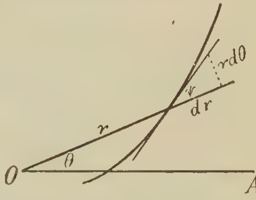


FIG. 133.

This is readily remembered by means of the polar differential triangle of Fig. 133, in which the dotted perpendicular $r\,d\theta$ may be regarded as equal to a circular arc swung with a radius r through an angle $d\theta$. See also Fig. 49, p. 135.

In (46) $dr/d\theta$ is the rate of change of r per *radian*. If for any reason we use the rate per degree, we must multiply the right member of (46) by $\pi/180$, or, .017453, approx.

Ex. I. Find ψ for the circle $r = 10 \cos \theta$ at $\theta = 45°$.

In the radian system: $dr/d\theta = -10 \sin \theta$.

$$\therefore \qquad \tan \psi = -\frac{10 \cos \theta}{10 \sin \theta} = -\text{ctn}\ \theta, \ = -1 \text{ at } \theta = 45°.$$

This gives $\psi = 135°$, which means that the circle (as θ increases through 45°) crosses its radius vector (directed outward) at an angle of 135°. This can be verified directly from Fig. 14, page 23.

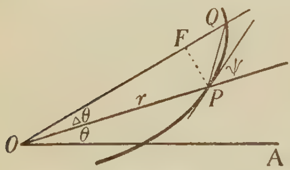


Fig. 134.

§ 230. Proof of tan ψ Formula. In Fig. 134, ψ is the limit of $\angle FQP$ as $\Delta\theta \to 0$. PF is drawn perpendicular to OQ.

$$OQ = r + \Delta r, \quad OF = r \cos \Delta\theta, \quad PF = r \sin \Delta\theta.$$

$$\therefore \qquad \tan FQP = \frac{PF}{OQ - OF} = \frac{r \sin \Delta\theta}{r + \Delta r - r \cos \Delta\theta}. \tag{47}$$

In the Maclaurin series for $\sin x$ and $\cos x$ (§ 146), if we replace x by $\Delta\theta$, we may write (47) thus:

$$\tan FQP = \frac{r\left(\Delta\theta - \dfrac{\Delta\theta^3}{3\,!} + \cdots\right)}{r + \Delta r - r\left(1 - \dfrac{\Delta\theta^2}{2\,!} + \cdots\right)}. \tag{48}$$

Dividing numerator and denominator by $\Delta\theta$, and simplifying:

$$\tan FPQ = \frac{r\left(1 - \dfrac{\Delta\theta^2}{3\,!} + \cdots\right)}{\dfrac{\Delta r}{\Delta\theta} + r\left(\dfrac{\Delta\theta}{2\,!} - \cdots\right)}. \tag{49}$$

Now let $\Delta\theta \to 0$. The numerator in (49) approaches $r(1)$, and the denominator approaches $dr/d\theta$. Hence

$$\tan \psi = \frac{r}{\dfrac{dr}{d\theta}}. \qquad \text{Q. E. D.}$$

§ 231. Equiangular Spiral and the *C. I. L.* The curve

$$r = ae^{k\theta} \tag{50}$$

is a spiral crossing all its radii vectores at a fixed angle. (Ex. 8, p. 386.) It is called the "equiangular spiral."

But (50) is the formula for the *Compound Interest Law*, with r and θ as the variables. Hence, if any quantity varying according to the *C. I. L.* be plotted on polar paper, the graph will cross all radial lines at the same angle. This fact is sometimes useful as a test for a *C. I. L.*, though the simplest test is usually that given in *Intro.*, § 175.

§ 232. Angle between Curves (Polar Coördinates). Where two curves cross, the angle $K_{1,2}$ between them is the difference of the angles ψ_1, ψ_2 at which they cross their common radius vector. (Verify by a figure.) Since $K_{1,2} = \psi_1 - \psi_2$:

$$\therefore \qquad \tan K_{1,2} = \frac{\tan \psi_1 - \tan \psi_2}{1 + \tan \psi_1 \tan \psi_2}. \tag{51}$$

To use this formula, first find the point of intersection and substitute in (46). Get each $\tan \psi$ numerically before using (51).

Caution. Curves may meet at the origin without their equations showing a simultaneous solution. This is because the polar coördinates of the origin are ambiguous: while r must be zero, θ may have any value. Two curves may enter at different angles.

Ex. I. Find the angle between $r = \sin \theta$ and $r = \cos \theta$ at each intersection.

(A) *Solving simultaneously:* $\theta = 45°$ or $225°$. (Only one point, however.) By (46):

$$\tan \psi_1 = \frac{\sin \theta}{\cos \theta} = 1, \qquad \tan \psi_2 = \frac{\cos \theta}{-\sin \theta} = -1.$$

Using these numerical values in (51) gives for the angle of crossing:

$$\tan K_{1,2} = \frac{1-(-1)}{1+(1)(-1)} = \infty, \quad \therefore K_{1,2} = 90°. \tag{52}$$

(B) *Testing the origin.* Both curves enter; for $r = 0$ in the first equation when $\theta = 0°$ and in the second when $\theta = 90°$. Hence the curves intersect, and their angle is again 90°.

EXERCISES

1. Find ψ for the following curves at the points specified:

(a) $r=2\theta$, at $\theta=.2, 1, 5$; (b) $r=\sin 3\theta$, at $\theta=\pi/6$;
(c) $r=2/\theta$, at $\theta=.2, 1, 5$; (d) $r=\cos 2\theta$, at $\theta=\pi/6$;
(e) $r=\theta^2$, at $\theta=.2, 1, 5$; (f) $r=a(1-\cos\theta)$, at $\theta=\pi/2$.

2. (a)–(f). In Ex. 1 (a)–(f), what angle does the tangent line in each case make with the polar axis, $\theta=0$?

3. The path of a comet is $r=a\sec^2(\theta/2)$. Describe its direction of motion when $\theta=\pi/2$.

4. The same as Ex. 3 for a planet: $r=k/(1+.016\cos\theta)$.

5. Plot $Q=2e^{.4t}$ on polar paper, taking t as 0, 1, $\cdots$, 5 (radians). Measure ψ at $t=\pi/3, \pi/2, \pi$. Check by (46).

6. Plot $Q=2e^{.4t}$ on rectangular paper, with Q vertical. Measure the subtangent, at $t=1, 2, 4$. Check by (37).

7. (a), (b). Like Ex. 5, 6, respectively, for $Q=30e^{-.5t}$.

8. Prove that ψ is constant for the curve (50); and that, conversely, whenever ψ is constant, r must vary according to (50).

9. Along an electric power line the voltage of the main wave varied with the distance (x mi.) as in Table I. Plot, using an angle of 12° for 100 mi. Did v decrease at an approximately constant percentage rate?

TABLE I

x	v
0	133000
500	109000
1000	90000
1500	74000
2000	61000
2500	50000

10. By geometry, why must the curves in Ex. I, p. 385, form the same angle at both intersections?

11. Find the angle formed by each following pair of curves at each intersection:

(a) $r=a\theta$, $r=a/\theta$; (b) $r=\sin 2\theta$, $r=\cos 2\theta$;
(c) $r=2^\theta$, $r=16(2^{-\theta})$; (d) $r=a(1+\cos\theta)$, $r=a(1-\cos\theta)$;
(e) the two curves in Fig. 67, p. 198.

§ 233. Polar S_T and Polar S_N.

The polar subtangent and polar subnormal are measured from the origin O to the tangent and normal, respectively, *along a line perpendicular to the radius vector OP.* (Fig. 135.)

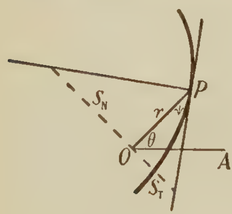


FIG. 135.

$$\frac{S_T}{r}=\tan\psi, \qquad \frac{S_N}{r}=\operatorname{ctn}\psi.$$

Substituting for $\tan\psi$ by (46) and for $\operatorname{ctn}\psi$ the reciprocal:

$$S_T = r\tan\psi = \frac{r^2}{\dfrac{dr}{d\theta}}, \qquad S_N = r\operatorname{ctn}\psi = \frac{dr}{d\theta}. \tag{53}$$

It is best not to memorize these formulas but to derive them, as here, when needed. The important thing to remember is what the polar S_T and S_N are, — where they are located.

§ 234. **Asymptotes.** To ascertain whether a polar curve recedes to infinity in any direction, and if so whether it approaches an asymptote, we may proceed as follows:

(1) See from the equation of the curve whether $r = \infty$ when θ has some value θ_1.

(2) See whether the polar subtangent S_T approaches some fixed value k as $\theta \to \theta_1$. If so, there is an asymptote at a distance from the origin equal to k numerically, and running in the direction $\theta = \theta_1$. (If k is negative, the asymptote is to the left, looking in the θ_1 direction.)

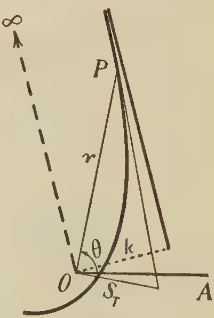


FIG. 136.

§ 235. **Asymptotes in Rectangular Coördinates.** There is also a systematic method for discovering asymptotes when the equation of a curve is given in rectangular form. We discuss here algebraic cases only.

(A) *Horizontal Asymptotes.* Let the equation be rationalized, cleared of fractions, and arranged in descending powers of x:

$$f_0x^n + f_1x^{n-1} + f_2x^{n-2} + \cdots + f_n = 0. \tag{54}$$

Here $f_0, f_1, \cdots, f_n$ are constants or functions of y. For a horizontal asymptote, $y = a$, the leading coefficient f_0 must be zero at $y = a$. And conversely.

Proof. Divide (54) by x^n:

$$f_0 + \frac{f_1}{x} + \frac{f_2}{x^2} + \cdots + \frac{f_n}{x^n} = 0. \tag{55}$$

This holds at all points on the curve. If, now, $y \to a$ as $x \to \infty$, then each fraction in (55) approaches zero, and so must f_0. Thus the substitution of a for y in f_0 must make $f_0 = 0$. Conversely, if $f_0 = 0$ when $y = a$, (55) is satisfied by $y = a$, $x = \infty$.

(B) *Vertical Asymptotes.* Likewise, a vertical asymptote is located by finding a value $x = a$, — if any exists, — which will reduce to zero the coefficient of the highest power of y in the equation.

(C) *Oblique Asymptotes.* Suppose that a curve

$$f(x, y) = 0 \tag{56}$$

has an asymptote $y = lx + b$. Then if we let $y = lx + v$ at any point of the curve, we know that $v \to b$ as $x \to \infty$. Hence, if we were plotting v as a function of x, the graph would have a horizontal asymptote $v = b$. By (A) above, the condition for this is that, in the v-equation, the coefficient of the highest power of actually present shall become zero when $v = b$. If the highest power apparently present does not involve v, it must somehow vanish if the asymptote is to exist. This will determine l.

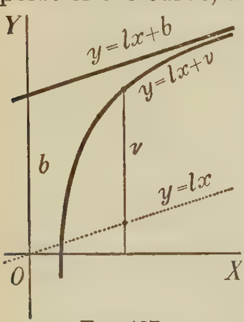


Fig. 137.

In practice, we substitute at the start $y = lx + b$, and put the highest coefficients equal to zero, — as just mentioned.

Ex. I. $4x^3 + 8x^2 - xy^2 - 6y^2 = 0.$ (57)

(A) The coefficient of x^3 is 4, which cannot vanish. There is no horizontal asymptote.

(B) The highest power of y is y^2, whose coefficient $-x-6$ vanishes at $x = -6$. There is a vertical asymptote, $x = -6$.

(C) Putting $y=lx+b$ in (57) and rearranging:

$$x^3(4-l^2)+x^2(8-2\,lb-6\,l^2)+x(\cdots)+(\cdots)=0.$$

Putting $4-l^2=0$ and $8-2\,lb-6\,l^2=0$ gives

$$l=\pm 2, \qquad b=-8/l=\mp 4. \tag{58}$$

Thus we have two oblique asymptotes:

$$y=2\,x-4, \qquad y=-2\,x+4. \tag{59}$$

The curve could be plotted by points easily, since (57) can be solved for y.

EXERCISES

1. Find the polar subtangent and subnormal of each following curve at the point specified:

(*a*) $r=a\theta,\ \theta=2$;
(*b*) $r=a/\theta,\ \theta=2$;
(*c*) $r=a^\theta,\ \theta=0$;
(*d*) $r=a\cos\theta,\ \theta=\pi/6$;
(*e*) $r=\sin 2\,\theta,\ \theta=\pi/4$;
(*f*) $r=1-\cos\theta,\ \theta=0$.

2. Do any of the curves in Ex. 1 recede to infinity? Do any have asymptotes?

3. Test each following curve for asymptotes:

(*a*) $r=5+2\sec\theta$,
(*b*) $r=a+b\sec\theta$,
(*c*) $r^2(\theta-1)=a^2$,
(*d*) $r=\sec 2\,\theta+\tan 2\,\theta$,
(*e*) $r^2=4\sec 2\,\theta$,
(*f*) $r=2\sec 2\,\theta$,
(*g*) $r=10\tan\theta\sec\theta$,
(*h*) $r=2\csc 3\,\theta$,
(*i*) $r=\dfrac{\sin^2\theta}{\cos\theta\cos 2\,\theta}$,
(*j*) $r=\dfrac{3\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}$,
(*k*) $r=\dfrac{a}{1-2\cos\theta}$,
(*l*) $r=\dfrac{a}{1-\cos\theta}$,
(*m*) $y^2-4\,x^2=36$,
(*n*) $x^2+4\,x-y^2=0$,
(*o*) $y^3-8\,x^3=4\,xy$,
(*p*) $x^3-xy^2+y^2=0$,
(*q*) $x^3+y^3=2\,ax^2$,
(*r*) $x^3-y^3-x^2+2\,y^2=0$,
(*s*) $xy^2-y^2-x^2=0$,
(*t*) $x^2y^2-y^4-a^2x^2+4\,a^2y^2=0$.

§ 236. Singular Points. The slope of a curve

$$f(x,\ y)=0 \tag{60}$$

can ordinarily be found by a single differentiation:

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0. \tag{61}$$

For (61) usually gives dy/dx as $-(\partial f/\partial x)\div(\partial f/\partial y)$.

If, however, these partial derivatives both happen to be zero at some special point of a curve, (61) does not define dy/dx. But we may differentiate further; and the next derived equation will in general give two values for dy/dx, — unless its coefficients also are zero. In the latter case, we differentiate yet again; and usually get three values for dy/dx. And so on.

When there are two real distinct values for dy/dx, the curve has two different slopes and two tangent lines. (Fig. 138.) Similarly when there are n distinct values of dy/dx. When there are two equal values, two branches of the curve come together with a common tangent. When there are two imaginary values, the curve does not exist near the point. The latter is therefore isolated, and belongs to the curve only in that its coördinates satisfy the equation.

Fig. 138.

A, *Node* or double point; *B*, *Salient point;* *C*, *Cusp* (common type); *D*, *Point of osculation;* *E*, *Conjugate point.*

Sometimes both branches of a curve lie on the same side of the common tangent at a cusp or point of osculation.

A salient point is possible only when the equation involves transcendental or irrational algebraic functions of special kinds. And, for such an equation, a curve sometimes has an "End Point," — where a single branch stops, because of imaginary values beyond.

A curve is said to have a "Singular Point" where any of the foregoing geometric peculiarities exists.

Singular points occur not only where $\partial f/\partial x$ and $\partial f/\partial y$ are both zero; but also (sometimes) where one of these is indeterminate, or one or both infinite. And, for some cases of multiple-valued functions, a cusp may occur at a point where one partial derivative alone is zero, — despite the fact that the slope is uniquely determined there by (61).

Ex. I. Find the slope of $4x^3+8x^2-xy^2-6y^2=0$ at $(0, 0)$.

Differentiating once:

$$(12x^2+16x-y^2)-(2xy+12y)\frac{dy}{dx}=0. \qquad (62)$$

At $(0, 0)$ both parentheses vanish. Differentiating again:

$$\left(24x+16-2y\frac{dy}{dx}\right)-(2xy+12y)\frac{d^2y}{dx^2} -\left(2x\frac{dy}{dx}+2y+12\frac{dy}{dx}\right)\frac{dy}{dx}=0.$$

At $(0, 0)$ this reduces to

$$16-12\left(\frac{dy}{dx}\right)^2=0. \qquad (63)$$

Hence $dy/dx=\pm\sqrt{4/3}$. There is a node at the origin.

This may be verified by plotting. (Cf. Ex. 3 *b*, p. 393.)

Ex. II. Has the above curve other singular points?

Since the equation is algebraic and rational, both partial derivatives must vanish at any such point:

$$12x^2+16x-y^2=0, \qquad 2xy+12y=0. \qquad (64)$$

The second equation requires $x=-6$ or $y=0$. There is no point on the curve for which $x=-6$. If $y=0$, the first equation of (64) requires $x=0$, or $-4/3$. But $(-4/3, 0)$ is not on the curve either. Hence $(0, 0)$ is the only singular point.

§ **237. Curve Tracing.** Some curves can be plotted easily by merely calculating points: substituting values for x or y, or r or θ; or cutting the curve by lines $y=lx$ through the origin, or using other parametric methods.

Time can often be saved if we supplement the calculation of points by finding the slope at various points (especially intersections with the axes), and by making tests for asymptotes, singular points, and horizontal or vertical tangents.

The tangent is horizontal (or $dy/dx=0$) if $\partial f/\partial x=0$ while $\partial f/\partial y\neq 0$. It is vertical if only $\partial f/\partial y=0$.

Ex. I. Trace the curve $x^3+9x^2=y^3+4y^2$.

Meets the axes:	$(0, 0)$, $(-9, 0)$, $(0, -4)$.
Slopes there:	$\pm 3/2$, ∞, 0.
Asymptote:	$y=x+\frac{5}{3}$.
Singular point:	$(0, 0)$, found above; no others.
Tangents horizontal:	$(-6, 3.74)$; and $(0, -4)$ above.
Tangents vertical:	$(.97, -\frac{8}{3})$, $(-1.10, -\frac{8}{3})$, $(-8.88, -\frac{8}{3})$; and $(-9, 0)$, found above.

Further points, if needed, can be found by cutting the curve by lines $y=lx$. The curve is shown in Fig. 139.

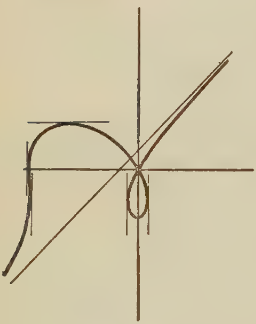


FIG. 139.

In locating the horizontal and vertical tangents, after putting $\partial f/\partial x=0$ and finding x [or $\partial f/\partial y=0$ and finding y], it is necessary to solve a cubic equation approximately for the values of the other coördinate.

§ 238. Remarks on Chapter IX. The formulas and processes of analytic geometry have two main kinds of uses: (1) To make investigations concerning curves whose equations are known, or concerning given sets of lines and points; and (2) To derive equations for curves defined by some geometric or locus property.

Sometimes such a property leads only to a differential equation, which must be solved to get the equation of the curve. (Cf. Ex. 9, p. 380, Ex. 8, p. 386.)

It is highly desirable to review the basic formulas and methods frequently, to keep them fresh in mind.

We proceed next to apply the various processes to a study of the most important class of curves known.

EXERCISES

1. Find the slopes of the following curves at (0, 0):

(*a*) $x^4 - 5x^3 - y^2 = 0$, (*b*) $x^3 + xy^2 + x^2 - 4y^2 = 0$,

(*c*) $x^4 + 4x^2y - 4y^3 = 0$, (*d*) $x^3 - xy^2 + 5x^2 + 3y^2 = 0$.

2. Plot by points from $x = -3$ to $x = 6$, showing any asymptotes:

(*a*) $y^2 = \dfrac{4x^2}{(x-1)(x-2)}$, (*b*) $y^2 = \dfrac{(x-1)(x-2)^2}{x-3}$,

(*c*) $x^3 - xy^2 = 8$, (*d*) $xy^2 - y^2 = x$.

3. Plot by points, checking the slope at (0, 0):

(*a*) $y^2 = x^4$, (*b*) $4x^3 + 8x^2 - xy^2 - 6y^2 = 0$.

4. Plot by points from $x = -4$ to $x = 4$: $y = 5 + \sqrt{x^2(x+4)}$.

[Observe that, when x is negative, this is not the same as $y = 5 + x\sqrt{x+4}$.] Note the salient point at $x = 0$ and the end point at $x = -4$. Are $\partial f/\partial x$ and $\partial f/\partial y$ zero at either?

5. Like Ex. 4 for $y = 5 - \sqrt{x^2(x+4)}$. Also show the complete locus of $(y-5)^2 = x^2(x+4)$.

6. Plot by points from $x = -4$ to $x = 4$: $y(1 + e^{\frac{1}{x}}) = x$. Define y at $x = 0$ so as to make it continuous there. Examine $\partial f/\partial x$ and $\partial f/\partial y$.

7. Plot $y = \dfrac{1}{\log x}$ from $x = 0$ to $x = .9$, defining y suitably at $x = 0$. How about negative values of x?

8. Verify the information listed in Ex. I, § 237, and thus check Fig. 139.

9. Make any helpful tests and plot the locus:

(*a*) $y^2 = x^3 - x^4$, (*b*) $x^3 - 2x^2y + xy^2 - y = 0$,

(*c*) $x^4 + 8x^2y - 4y^3 = 0$, (*d*) $x^4 - xy^3 + y^3 - 2x^2 = 0$,

(*e*) $y^5 + 2x^4 - xy^2 = 0$, (*f*) $x^2y + y^3 + x^2 - xy = 0$.

10. A point (x, y) moves so as to be always equidistant from (7, 1) and the line $3x + 4y = 10$. What must the path be? Derive its equation.

11. Find the center of the circle inscribed in the triangle whose vertices are (0, 0), (300, 125), (168, 224).

12. Draw lines $y = 2x/k$, $y = 4k$, for several values of k. What is the locus of the intersection, apparently? Give a proof.

13. Find the angle between $r = e^{\theta}$, $r = 4e^{-\theta}$, where they cross.

14. Find S_T and S_N for $y^2 = 20x$ at $x = 10$.

15. Find a curve having $S_T = y^2$ at every point.

CHAPTER X

THE CONICS

PART I. DEFINITION AND CLASSIFICATION

§ 239. Definition and Equation. The locus of a point P, whose distances from a fixed point F and a fixed line DD' have a constant ratio, is called a *conic*. The ratio ($e = FP/DP$) is called the "eccentricity," F a "focus," and DD' a "directrix."

Choosing axes as in Fig. 140, we have

$$\frac{\sqrt{(x-m)^2+y^2}}{x}=e.$$

$$\therefore \qquad \boldsymbol{x^2(1-e^2)-2\,mx+y^2+m^2=0.} \qquad (1)$$

The shape and position of the conic depends upon the parameters m and e; that is, upon the distance of the focus from the directrix, and upon the constant ratio.

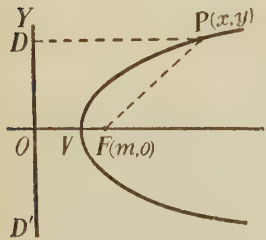


FIG. 140.

Giving m a negative value merely reflects the conic in the Y-axis.

If $m=0$, the focus lies on the directrix, and (1) becomes

$$x^2(1-e^2)+y^2=0. \qquad (2)$$

This covers several special cases. (Cf. Ex. 4, p. 397.)

§ 240. $e=1$: The Parabola. When $e=1$, (1) becomes

$$y^2=2\,mx-m^2=2\,m(x-m/2). \qquad (3)$$

This represents a parabola, — as it should, since the distances FP and DP are now equal. Also, F and DD' fit the earlier definition of focus and directrix.

In (3), $2m$ is the $4p$ of our earlier notation, and the curve is displaced $m/2$ units to the right, making its vertex $V(m/2, 0)$, or $(p, 0)$.

The smaller the value of m, the nearer the vertex V is to the directrix DD', and the narrower the curve.

§ **241.** $\boldsymbol{e<1}$: **The Ellipse.** If $e<1$, $(1-e^2)$ is positive. Completing the square in (1), and dividing through by the resulting right member gives [Ex. 5, p. 397]:

$$\frac{\left(x-\frac{m}{1-e^2}\right)^2}{\frac{m^2e^2}{(1-e^2)^2}}+\frac{y^2}{\frac{m^2e^2}{1-e^2}}=1. \tag{4}$$

Here both denominators are positive and the first is the larger. Hence (4) represents an ellipse with

$$a=\frac{me}{1-e^2},\quad b=\frac{me}{\sqrt{1-e^2}},\quad \text{center } \left(\frac{m}{1-e^2}, 0\right). \tag{5}$$

Moreover, $FC=OC-OF=m/(1-e^2)-m=me^2/(1-e^2)$. From this and from (5) it easily follows that $\overline{FC}^2=a^2-b^2$. In our former notation $a^2-b^2=c^2$. (*Intro.*, § 211.) Hence F, a focus of the conic, is also a focus of the ellipse as formerly defined.

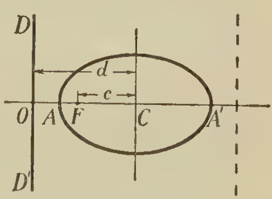


Fig. 141.

We have previously seen that the ellipse is symmetrical and has two foci. Evidently it must have a second directrix (dotted in Fig. 141), and must bear the same relation to this and to the second focus as to DD' and F.

Observe further that

$$FC\times OC=\frac{m^2e^2}{(1-e^2)^2}=a^2, \quad \text{or} \quad a^2=cd. \tag{6}$$

That is, *the major semi-axis* **a** *is a mean proportional between the distances* **c** *and* **d** *of the center from either focus and from the directrix.*

The ellipse will be studied further presently.

§ **242.** $e>1$: **The Hyperbola.** If $e>1$, $(1-e^2)$ is negative; and (4) is then better written:

$$\frac{\left(x+\frac{m}{e^2-1}\right)^2}{\frac{m^2e^2}{(e^2-1)^2}}-\frac{y^2}{\frac{m^2e^2}{e^2-1}}=1. \tag{7}$$

Hence the conic is an hyperbola, with $a=me/(e^2-1)$, $b=me/\sqrt{e^2-1}$, and center $[-m/(e^2-1), 0]$.

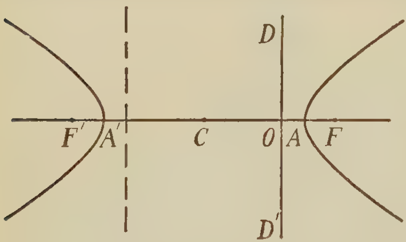


FIG. 142.

Moreover, CF or $(CO+OF)$ now reduces to $me^2/(e^2-1)$, whence we readily find

$$\overline{CF}^2=a^2+b^2; \tag{8}$$

$$CF\times CO=a^2. \tag{9}$$

These equations show that the foci of the conic are those of the hyperbola, as defined in *Intro.*, § 216; and that the transverse semi-axis a is a mean proportional between c and d, — as for the ellipse.

§ **243. Summary.** A conic is in general an ellipse, parabola, or hyperbola, according as its eccentricity e is less than, equal to, or greater than 1. But if $m=0$, *i.e.*, if the focus lies on the directrix, the conic degenerates into a simpler locus. (See Ex. 4 below.)

EXERCISES

1. Derive in detail the equation of a conic of eccentricity $\frac{1}{2}$, with (3, 0) as a focus and the Y-axis as a directrix. Compare your equation with (1). Also complete the square and compare with (4). What curve? What center? Semi-axes?

2. Like Ex. 1 for a conic with the same focus and directrix, but with eccentricity 2.

3. Derive, and reduce to a standard form, the equation of a conic with the Y-axis as a directrix, and with

(*a*) Focus (16, 0), $e=3/5$; (*b*) Focus (12, 0), $e=3$;
(*c*) Focus (5, 0), $e=1$; (*d*) Focus (12, 0), $e=0$.

Comment on the result in (*d*).

4. Derive the equation of a conic having (0, 0) as a focus, and the Y-axis as a directrix, if $e=5/3$. Also if $e=1$. Also if $e=4/5$. Show the reasonableness of each result.

5. Complete the square in (1) and obtain (4), p. 395.

6. Find from (4) the points where the conic in Fig. 141 meets the X-axis. Show that their midpoint is the center as given by (5).

7. As outlined in § 241 show that $a^2-b^2=\overline{FC}^2$.

8. Find the equation of a conic with $e=5/4$, focus (0, 9), and the X-axis as directrix.

9. Find the equation of a conic with $e=\frac{1}{2}$, focus (3, 3), and the line $x+y=0$ as directrix. Then rotate the axes 45° and simplify.

10. The same as Ex. 9 if $e=1$.

§ 244. The Standard Ellipse. As defined in *Intro.*, § 210, an ellipse is the locus of a point whose distances from two fixed points or "foci" have a constant sum $2a$. (That sum $2a$ is also the length of the major axis.) We may now say further that an ellipse is a *conic whose eccentricity is less than unity.* That is, the distance of any point on the ellipse from either focus is a certain constant fraction ($e<1$) of the distance from the corresponding directrix.

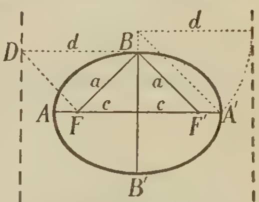


FIG. 143.

The foci lie on the major axis. They can be located geometrically by using the fact that the distance of each from either end of the minor axis is a. The directrices, perpendicular to the major axis, can be located by using the fact that a is a mean proportional between c and d, or $a^2=cd$. Two possible geometrical constructions of d are shown in Fig. 143.

(1) A perpendicular to BF at F will meet the tangent BD at a distance d from B and hence on a directrix. (2) The parallel to $F'B$ from A' will meet $B'B$, produced, at a distance d from A'.

PROOF: In each case similar triangles are formed, in which

$$\frac{c}{a}=\frac{a}{d}, \qquad \text{or } a^2=cd,$$

which shows the correctness of the construction.

The eccentricity e is the ratio $FB/DB=a/d$, or c/a.

All these statements apply, no matter in what direction the ellipse is turned.

Ex. I. Find the foci, directrices, and eccentricity of

$$\frac{x^2}{36}+\frac{y^2}{100}=1.$$

Here $a=10$, $b=6$; the major axis is vertical, and the foci are $(0, 8)$, $(0, -8)$. Verify by drawing a figure. The eccentricity is $e=c/a=4/5$. Also $a^2=cd$; hence $d=12.5$. The directrices are the lines $y=\pm 12.5$.

The ellipse could be drawn with a string 20 units long whose ends are fastened 16 units apart, — or with a loop 36 units long.

§ **245. The Standard Hyperbola.** The type equation of the hyperbola as previously defined and studied is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{10}$$

Its asymptotes are the lines $y=\pm\frac{b}{a}x$; or, together:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0. \tag{11}$$

They are drawn through $(0, 0)$ and the points $(a, \pm b)$. The foci $(\pm c, 0)$ are located by the circle through the latter points with center $(0, 0)$. For $c^2=a^2+b^2$.

In the hyperbola a may be less than b. But, — as for the ellipse, — a denotes the distance from the center to the

"vertices," *i.e.*, the points where the curve crosses its "transverse axis" (the line through the foci).

The eccentricity is $e=c/a(>1)$. By (9) the distance d of either directrix from the center is again given by $cd=a^2$. As $c>a$, evidently $d<a$. The directrices D_1D_2 and $D'_1D'_2$ thus lie between the two branches of the curve.

Geometrically, the directrices can be located by cutting the asymptotes by an arc of radius a with O as center.

For, in the similar right triangles of Fig. 144:

$$\frac{d}{a}=\frac{a}{c}, \quad \text{the correct relation.}$$

§ **246. The Conjugate Hyperbola.** The line CC' through the center O, perpendicular to the transverse axis FF' of an hyperbola, is called the conjugate axis. Another hyperbola, with foci on CC' at the same distance c from O, and with the same asymptotes, is called the *conjugate hyperbola* of the first, — shown dotted in Fig. 144. Its equation, in terms of a and b for the first, is

$$y^2/b^2-x^2/a^2=1,$$

or

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1. \qquad (12)$$

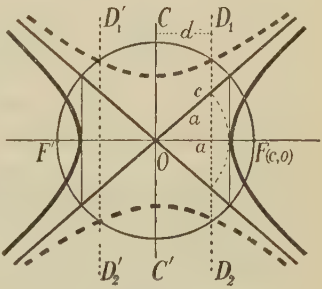


FIG. 144.

An hyperbola, its asymptotes, and its conjugate are, therefore, all represented by the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=k, \qquad (13)$$

if we give k the respective values 1, 0, -1.

EXERCISES

1. How was the hyperbola defined in *Intro.*, § 215? What does the statement that it is a conic with $e>1$ mean?

2. What foci and directrices (also asymptotes, if any) has each following curve?

(a) $9x^2+25y^2=225$, (b) $25x^2+16y^2=400$,
(c) $9x^2-16y^2=144$, (d) $16y^2-9x^2=144$.

3. The same as Ex. 2 for each following curve:

(a) $\dfrac{(x-4)^2}{169}+\dfrac{(y+6)^2}{25}=1$, (b) $\dfrac{(x+2)^2}{36}-\dfrac{(y-5)^2}{64}=1$,
(c) $16x^2+25y^2-100y=300$, (d) $x^2-y^2-4x=4$.

4. Plot $9x^2 + 25y^2 = 900$, finding a number of points accurately by calculation or construction. Also locate the foci and directrices geometrically, and check by numerical calculation.

5. Like Ex. 4 for $9x^2 - 16y^2 = 576$.

6. For a conic having each following pair of foci and directrices, find a and b; and then write the equation:

(a) $(\pm 16, 0)$, $x=\pm 25$; (b) $(\pm 25, 0)$, $x=\pm 16$;
(c) $(\pm 50, 0)$, $x=\pm 18$; (d) $(\pm 18, 0)$, $x=\pm 50$.

7. Find the foci and directrices of $\dfrac{x^2}{625}+\dfrac{y^2}{b^2}=1$, if $b=7$. Likewise if $b=15$, 20, 24. What happens if $b=25$?

8. What is the limiting form of an hyperbola if we let $a \to 0$ and always keep $b=a$? If $b=2a$?

§ 247. Limiting Forms. Besides the degenerate conics already noted, certain other limiting forms deserve mention. (See also Ex. 12–14, p. 403.)

(I) *The Circle.* In an ellipse, if b is nearly equal to a, c must be small and d large. Thus the circle is the limiting form of an ellipse, as $c \to 0$ and $d \to \infty$. (Cf. Ex. 7, above.) By the strict definition, § 239, a conic cannot be a circle. But we may say, in the limiting sense, that a circle is a conic of eccentricity zero, whose directrix lies at infinity. Studies of the ellipse cover the circle, on putting $b=a$.

(II) *The Parabola.* By § 239, if $e \to 1$ while DD' and F are held fixed, the ellipse approaches as its limiting form a parabola with the same directrix and focus.

(III) *Another Parabola.* Let us also find the limiting

form of an ellipse, if a focus F and the nearer vertex A be fixed while the other vertex recedes to infinity.

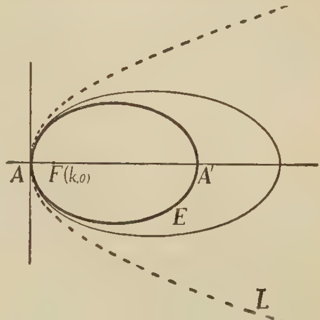


FIG. 145.

Taking the origin at A, the center is a units to the right and the equation of the ellipse is

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1.$$

Cleared of fractions and simplified,

$$b^2x^2-2\,b^2ax+a^2y^2=0. \qquad (14)$$

Since AF is to be constant, let $a-c=k$, or $c=a-k$. Also replace b^2 by a^2-c^2, or let $b^2=a^2-(a-k)^2=2\,ak-k^2$. Then (14) becomes

$$(2\,ak-k^2)x^2-2(2\,a^2k-ak^2)x+a^2y^2=0. \qquad (15)$$

Dividing through by a^2 and then letting $a \to \infty$, we find in the limit that all terms disappear, except: $-4\,kx+y^2=0$. That is, $y^2 = 4\,px$.

Thus the required limiting form is a parabola with the same vertex and focus, (0, 0) and (k, 0), respectively.

§ 248. Polar Equations. Equations of the conics in polar coördinates are much used, — especially in Astronomy. The pole is generally taken at a focus.

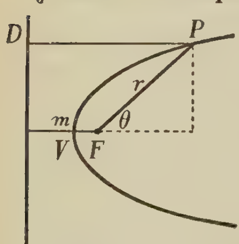


FIG. 146.

For any conic (Fig. 146), by definition, $FP=eDP$. That is,

$$r=e(m+\cos\theta).$$

$$\therefore \qquad r=\frac{me}{1-e\cos\theta}. \qquad (16)$$

This represents an ellipse, parabola, or hyperbola, on taking $e<1$, $e=1$, or $e>1$, respectively.

If Fig. 146 were reversed, with DD' to the right of F and the conic running to the left, the equation would come out (as in Ex. 1, p. 402):

$$r=\frac{me}{1+e\cos\theta}. \qquad (17)$$

Instead of m, other constants may be known; and special forms of (16) and (17) may be more convenient.

The Ellipse. From the value of a in § 241 we have at once $me = a(1-e^2)$. Hence (16) becomes

$$r = \frac{a(1-e^2)}{1-e\cos\theta}. \tag{18}$$

Likewise for (17), but with the denominator $1+e\cos\theta$.

The Parabola and Hyperbola. See Ex. 2–4 below.

Without memorizing any rules, the position of a conic represented by any of these polar equations can be determined by simply substituting $\theta = 0°$, and $\theta = 180°$.

Ex. I.
$$r = \frac{a(1-e^2)}{1+e\cos\theta}. \tag{19}$$

At $\theta = 0°$, $r = a(1-e)$; at $\theta = 180°$, $r = a(1+e)$.

The right-hand vertex is the closer to the origin; in other words, the origin or pole is at the right-hand focus.

Ex. II.
$$r = \frac{20}{1-\cos\theta}. \tag{20}$$

At $\theta = 0°$, $r = \infty$; at $\theta = 180°$, $r = 10$.

The vertex is to the left of the focus and the curve (a parabola) opens toward the right.

§ **249. Latus Rectum.** The chord through a focus of a conic, perpendicular to the transverse or principal axis, is called the *latus rectum*. Its length is twice the value of r at $\theta = 90°$ in (16), viz. $2r = 2me$.

No formula need be memorized. Simply substitute each time for θ or x in the polar or rectangular equation and solve for r or y. Then take $2r$ or $2y$. (Or, if the transverse axis be vertical, substitute for y and solve for x.)

EXERCISES

1. Derive equation (17) for the reversed conic.

2. What does (16) become for a parabola? [What is e? m?]

3. What form does (17) assume for a parabola? Show that it may be written also: $r = p \sec^2 (\theta/2)$.

4. What forms do (16) and (17) assume for the hyperbola?

5. Recognize the type of conic represented by each following equation, and determine its position as in Ex. I, II, § 248. Also find the length of the latus rectum; and draw roughly.

(*a*) $r = \dfrac{12}{1 - .5 \cos \theta}$, (*b*) $r = \dfrac{12}{1 - 2 \cos \theta}$, (*c*) $r = \dfrac{12}{1 + \cos \theta}$,

(*d*) $r = \dfrac{5}{1 + 4 \cos \theta}$, (*e*) $r = \dfrac{6}{1 + .2 \cos \theta}$, (*f*) $r = \dfrac{3}{1 - \cos \theta}$,

(*g*) $r = \dfrac{8}{2 + \cos \theta}$, (*h*) $r = \dfrac{7}{2 - 3 \cos \theta}$, (*i*) $r = \dfrac{4}{.5 - \cos \theta}$.

6. If the parabola $y^2 = 20\,x$ be moved, without rotation, until its focus is at the origin, what will its equation be? Transform to polar coördinates.

7. Like Ex. 6 for the conic in Fig. 140. Compare the final equation with (16).

8. If every radius drawn from the focus to a conic be bisected, what will be the locus of the midpoint? Experiment, and give a proof.

9. Shift each of the following, to bring the left-hand focus to the origin; and transform to the polar equation:

(*a*) $9\,x^2 + 25\,y^2 = 3600$, (*b*) $16\,x^2 - 9\,y^2 = 3600$.

10. (*a*), (*b*). Like Ex. 9 for the right-hand focus.

11. Find the length of the latus rectum of each following conic:

(*a*) $16\,x^2 + 25\,y^2 = 400$, (*b*) $25\,x^2 + 9\,y^2 = 225$,
(*c*) $16\,x^2 - 9\,y^2 = 144$, (*d*) $16\,y^2 - 9\,x^2 = 3600$,
(*e*) $y^2 - 12\,x = 0$, (*f*) $x^2 + 32\,y = 0$.

12. In Fig. 145 replace the ellipse by an hyperbola. Let a focus F and the nearer vertex A be fixed, while the other vertex recedes to infinity. Show that the limiting form is the same parabola as in Fig. 145.

13. In Fig. 140 hold F and DD' fixed; and determine the limiting form of the conic as $e \to 0$. Also as $e \to \infty$.

14. A parabola passes always through (0, 4) and (0, −4), while its vertex recedes to infinity along the X-axis. Show that the limiting form is a pair of parallel lines.

15. Reduce to standard forms, and draw roughly:

(*a*) $2\,x^2 + 3\,y^2 - 4\,x + 12\,y = 10$, (*b*) $x^2 + 8\,x - 2\,y = 0$,
(*c*) $x^2 - y^2 + 10\,x - 4\,y = 28$, (*d*) $x - y^2 - 6\,y = 0$.

§ 250. General Equation of the Second Degree. The basic rectangular equation of any conic, (1), p. 394, is of the second degree, — and this degree could not be changed by choosing a different origin or rotating the axes. (Ex. 13, p. 377.) We shall now show, conversely, that every equation of the second degree represents some conic, if it has any locus at all. (Cf. p. 4.)

(I) In any equation of the form

$$Ax^2+Cy^2+Dx+Ey+F=0, \tag{21}$$

we can complete the squares, — or square, if one be missing. This will give an equation involving "translaters" and recognizable as the equation of an ellipse, parabola, or other conic, — if it has a real locus.

(II) The general equation of the second degree is

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0. \tag{22}$$

Proceeding exactly as in Ex. II, § 221, we can rotate the axes through a suitable angle, and so transform (22) into an equation of the form (21) above, free from the product xy.

That is to say: Rotating through *any* angle ϕ will give a new equation whose xy term is

$$[-2\,A \cos\phi \sin\phi + B(\cos^2\phi - \sin^2\phi) + 2\,C \sin\phi \cos\phi]xy;$$

and the coefficient of xy here will be zero if

$$B\tan^2\phi+2(A-C)\tan\phi-B=0. \tag{23}$$

$$\therefore \qquad \tan\phi=\frac{-(A-C)\pm\sqrt{(A-C)^2+B^2}}{B}. \tag{24}$$

As the quantity under the radical is necessarily positive, (24) gives real values for $\tan\phi$. Hence ϕ can be found.

The rotation may possibly remove the x^2 term in (22), or the y^2 term, as well as the xy term, but not all three, as that would change the degree of the equation. Thus the modified equation is of the type (21). Since the latter represents a conic, if any real locus, so must (22).

§ 251. Recognizing the Curve Directly. The nature of a conic given by an equation of the form (22) with numerical coefficients can be recognized, and the curve plotted by points, without reducing the equation to a standard form by rotation, etc. The method is based upon certain differences between the various conics.

(1) An *ellipse* is of limited extent. Substituting very great + or − values for x (or y) in its equation must make y (or x) imaginary.

(2) An *hyperbola* ultimately approaches its asymptotes closely. Hence y must be real for very great values of x, either + or −; and vice versa.

(3) A *parabola* whose axis is tilted must somewhere have a vertical tangent; and must lie entirely to the right of this tangent, or else entirely to the left. Thus, very great values of x will make y real in one direction (+ or −), and imaginary in the other.

Ex. I. $$x^2-2xy+y^2+6x+8y-12=0. \tag{25}$$
Here $$y^2+(8-2x)y+(x^2+6x-12)=0.$$

Solving by the formula for a quadratic equation $Ay^2+By+C=0$, and simplifying, we get

$$y=x-4\pm\sqrt{28-14x}. \tag{26}$$

Here y is real for large negative values of x, but imaginary for large positive values, — in fact, if $x>2$. The curve is a parabola lying to the left of the vertical line $x=2$.

Ex. II. $$6x^2-xy-2y^2-7x-7y-5=0. \tag{27}$$
Here $$2y^2+(x+7)y+(-6x^2+7x+5)=0.$$

The radical expression in the formula turns out to be $49x^2-42x+9$, a perfect square; and we find

$$y=\tfrac{3}{2}x-\tfrac{5}{2}, \quad \text{or } y=-2x-1. \tag{28}$$

The locus is a pair of intersecting straight lines. This is a degenerate hyperbola, — a conic with $m=0$.

§ 252. Inspection Test: B^2-4AC. By proceeding as in § 251, with the general equation

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0, \tag{29}$$

we can show that, aside from the possibility of degenerate cases, the nature of the conic represented by (29) can be determined by merely noting the sign of B^2-4AC:

B^2-4AC	$-$	0	$+$
Conic	ellipse	parabola	hyperbola

EXERCISES

1. Verify equation (26) above and plot the curve.

2. Verify (28) and plot the lines.

3. Determine as in § 251 the nature of the locus of each following equation. Also check by the test in § 252; and plot.

(*a*) $2x^2+9xy+3y^2=13$, (*b*) $x^2-2xy+2y^2=2$,
(*c*) $4x^2+5xy+y^2=9$, (*d*) $4x^2-12xy+9y^2=36$,
(*e*) $x^2+y^2-2xy+8x+8y+16=0$,
(*f*) $2x^2+xy-6y^2-7x-7y+5=0$.

4. Establish the inspection test of § 252.

5. Barring degenerate cases, what sort of conic will each following equation represent:

(*a*) $2x^2+5xy-3y^2=1$, (*b*) $x^2+xy+y^2=10$,
(*c*) $x^2+4xy+4y^2=x$, (*d*) $x^2+3xy+1=0$?

§ **253. Reducing the Equation.** By rotating the axes through a suitable angle ϕ, the general equation

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

can be reduced to one involving no xy term. (§ 250.) Let us carry out this operation in a few cases.

In the formula for the angle of rotation,

$$\tan\phi=-\left(\frac{A-C}{B}\right)\pm\sqrt{\left(\frac{A-C}{B}\right)^2+1}, \tag{30}$$

we may choose the $+$ sign, and thus use an acute angle ϕ. A check upon the work is to see that the xy term after the rotation actually does disappear.

Ex. I. $$5x^2+4xy+2y^2+9x-3y+10=0. \tag{31}$$
Here $A-C=3$, $B=4$:

$$\therefore\ \tan\phi=-\tfrac{3}{4}+\sqrt{(\tfrac{3}{4})^2+1}=\tfrac{1}{2}.$$

This gives $\sin\phi=1/\sqrt{5}$, $\cos\phi=2/\sqrt{5}$. Hence, by (24), § 221, the rotation is effected by replacing every

$$x \text{ by } \frac{2x-y}{\sqrt{5}}; \qquad y \text{ by } \frac{x+2y}{\sqrt{5}}.$$

Putting these expressions in (31), multiplying out, and reducing, we finally get:

$$6x^2+y^2+3\sqrt{5}\,x-3\sqrt{5}\,y+10=0. \tag{32}$$

There is no xy term left. Completing the squares:

$$6(x+\sqrt{5}/4)^2+(y-3\sqrt{5}/2)^2=25/8. \tag{33}$$

$$\therefore \qquad \frac{\left(x+\frac{\sqrt{5}}{4}\right)^2}{\frac{25}{48}}+\frac{\left(y-\frac{3\sqrt{5}}{2}\right)^2}{\frac{25}{8}}=1. \tag{34}$$

The locus is an ellipse with $a^2=25/8$, $b^2=25/48$, and center at $(-\sqrt{5}/4,\ 3\sqrt{5}/2)$, *referred to the new axes.* These are inclined at an angle $\tan^{-1}\frac{1}{2}$ [$=26°\ 34'$, approx.] to the old.

Check. Let us cut the curve by the line $y=-x$ (original coördinates). Putting $y=-x$ in (31) gives

$$3x^2+12x+10=0. \tag{35}$$

This has the roots $x=(-6\pm\sqrt{6})\div 3$; or -1.18 and -2.82, approx. These intersections are in accord with Fig. 147.

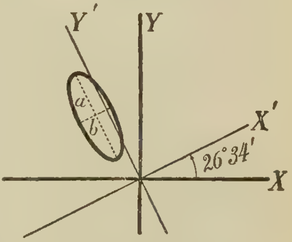


Fig. 147.

Ex. II. $$4x^2+12xy+9y^2-7x-4y+5=0. \tag{36}$$

$$(A-C)/B=-\tfrac{5}{12}, \qquad \tan\phi=\tfrac{5}{12}+\sqrt{(\tfrac{5}{12})^2+1}=\tfrac{3}{2}.$$

From this: $\sin\phi = 3/\sqrt{13}$, $\cos\phi = 2/\sqrt{13}$. Hence we replace x by $(2x-3y)/\sqrt{13}$, and y by $(3x+2y)/\sqrt{13}$. Upon reducing, this gives

$$13x^2 - 2\sqrt{13}\,x + \sqrt{13}\,y + 5 = 0,$$

or

$$\left(x - \frac{1}{\sqrt{13}}\right)^2 = -\frac{1}{\sqrt{13}}\left(y + \frac{4}{\sqrt{13}}\right). \tag{37}$$

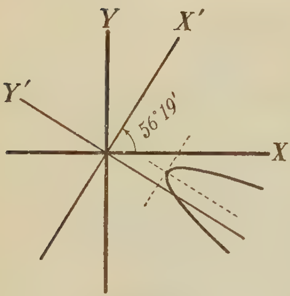


FIG. 148.

The locus is a parabola extending in the new negative Y-direction. (Fig. 148.) The new axes are inclined at an angle $\tan^{-1}\frac{3}{2}$ [$=56°\ 19'$, approx.] to the old.

There are other methods of reducing the general equation of the second degree. But these differ much in different cases, and the foregoing process is sufficient in all.

EXERCISES

1. Using the substitutions in Ex. I, § 253, verify the reduction of (31) to (34).

2. Likewise verify the reduction of (36) to (37).

3. Reduce each following equation to a standard form. Also plot the locus, showing its relation to the old and new axes. Check by the test of § 252, and by some convenient cutting line.

(*a*) $6x^2 - 4xy + 3y^2 + 4x + 8y - 18 = 0$,
(*b*) $16x^2 - 8xy + y^2 + 34x + 136y = 0$,
(*c*) $2x^2 + 7xy - 22y^2 - 30x - 60y = 0$,
(*d*) $10x^2 + 12xy + 5y^2 - 4x + 6y - 1 = 0$,
(*e*) $4x^2 - 5xy - 8y^2 + 31x + 38y + 280 = 0$,
(*f*) $x^2 + 6xy + 9y^2 - 4x - 12y = 0$,
(*g*) $4x^2 + 3xy + 26x + 12y + 22 = 0$.

4. Rationalize, reduce to a standard form, and plot: $\sqrt{x} + \sqrt{y} = \sqrt{a}$. (Have you encountered this equation previously?)

5. Plot the curve $2x^2 + 2y^2 - 8x - 12y + 5 = 0$.

PART II. GEOMETRIC PROPERTIES

§ 254. A Locus Problem. What is the locus of the midpoints of a set of parallel chords of an ellipse?

Experiment indicates a straight line segment through the center. (For a circle the locus is known to be such a line, perpendicular to the bisected chords.)

Let each chord have the equation

$$y=lx+k, \tag{38}$$

where l is the fixed slope and k changes from chord to chord. The ends of the chord (x_1, y_1), (x_2, y_2), may be found by solving (38) simultaneously with the equation of the ellipse, $b^2x^2+a^2y^2=a^2b^2$. Combining equations and collecting terms gives

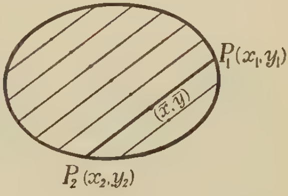


FIG. 149.

$$(l^2a^2+b^2)x^2+2\,a^2lkx+a^2(k^2-b^2)=0. \tag{39}$$

The two roots of this are x_1 and x_2, but we care only for the midpoint,

$$\bar{x}=\tfrac{1}{2}(x_1+x_2), \qquad \bar{y}=\tfrac{1}{2}(y_1+y_2).$$

In any quadratic equation $Ax^2+Bx+C=0$, the usual formula for the roots x_1 and x_2 shows that $x_1+x_2=-B/A$. In (39), $A=(l^2a^2+b^2)$ and $B=2\,a^2lk$.

$$\therefore \qquad \bar{x}=-\frac{a^2lk}{l^2a^2+b^2}. \tag{40}$$

Substituting this for x in (38) and simplifying gives

$$\bar{y}=\frac{b^2k}{l^2a^2+b^2}. \tag{41}$$

With k changing, (40) and (41) are parametric equations of the required locus. Eliminating k gives

$$y=-\frac{b^2}{a^2l}x. \tag{42}$$

This represents a straight line through the center (0, 0).

Remarks. (I) For an hyperbola, we should replace b^2 everywhere above by $-b^2$. Thus (42) would become

$$y = \frac{b^2}{a^2 l} x. \tag{43}$$

The locus is again a straight line through the center.

(II) For the parabola $y^2 = 4\,px$, the locus is found to be

$$y = \frac{2\,p}{l}. \tag{44}$$

§ **255. Diameters.** The locus of the midpoints of a complete set of parallel chords of a conic is called a *diameter*. In each case it is a straight line segment, — with one of the equations (42)–(44) above, if the axes are taken in the standard positions.

Let l_1 denote the slope of (42), the diameter of the ellipse. Then

$$l_1 = -\frac{b^2}{a^2 l}. \tag{45}$$

This is negative when the slope l of the chords is positive; and vice versa. (How about the hyperbola?)

Solving (45) for l:

$$l = -\frac{b^2}{a^2 l_1}. \tag{46}$$

This shows that a second diameter, of slope l, parallel to the original chords, would bisect all chords of slope l_1. In other words, *if one diameter bisects the chords parallel to a second diameter, then the second bisects the chords parallel to the first.* Two diameters so related are said to be "conjugate." The relation (45) or (46) between their slopes may be written also

$$l l_1 = -\frac{b^2}{a^2}. \tag{47}$$

A tangent at the extremity of a diameter is the limiting position of one of the parallel chords or secants. Hence *the tangent at the extremity of any diameter is parallel to the conjugate diameter.*

§ 256. Geometric Constructions. If we have given an ellipse, — the curve only, — and desire to determine geometrically the center, foci, directrices, etc., we may

(1) Find the center by constructing two diameters;

(2) Find the principal axes by cutting the ellipse with a concentric circle and using the symmetrical points of intersection;

(3) Find the foci by the usual method (§ 244);

(4) Locate the directrices as in § 244.

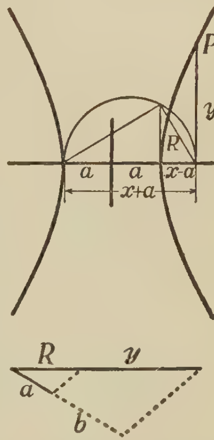


FIG. 150.

For an *hyperbola* the steps would be much the same as above except that, before starting (3), we must determine the asymptotes, or the distance b. From (10), p. 398,

$$y=\frac{b}{a}\sqrt{x^2-a^2}, \text{ or } \frac{b}{a}=\frac{y}{\sqrt{x^2-a^2}}, \quad (48)$$

for any point P on the upper half of the curve. Thus b is to a as y is to $\sqrt{x^2-a^2}$. Also, this radical [call it R] is the mean proportional between $x+a$ and $x-a$. The constructions in Fig. 150 will give first R and then b; after which we can draw the asymptotes, and find the foci and directrices, as in § 398.

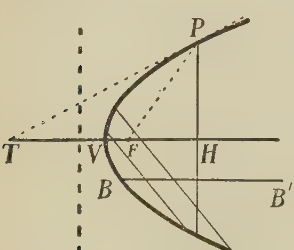


FIG. 151.

§ 257. The Case of the Parabola. Given a parabola we can determine its axis by constructing any diameter, drawing a chord perpendicular to it, and bisecting the latter perpendicularly. The tangent at any point P can then be drawn by extending the axis to T (Fig. 151), making $VT=HV$. The focus F can next be located by making $\angle TPF=\angle PTF$. [*Intro.*, p. 299, Ex. 10.] Finally, the

directrix can be drawn, since V is equidistant from it and from F.

EXERCISES

1. What is the formula for the roots x_1 and x_2 of the equation $Ax^2+Bx+C=0$? Verify that $x_1+x_2=-B/A$, as stated in § 254.

2. Draw an hyperbola approximately. Find experimentally the locus of the midpoints of a set of parallel chords, using both branches. Derive the equation (43), by the method used for (42).

3. In the drawing of Ex. 2, insert the conjugate hyperbola and more parallel chords meeting it. Do their midpoints seem to fall into line with the others? Test by deriving the equation of the new locus.

4. Like Ex. 2 for a parabola and equation (44).

5. Draw approximately an ellipse and the diameter bisecting two parallel (sloping) chords. Also draw the conjugate diameter and the tangent line at either end of the first diameter.

6. Like Ex. 5 for an hyperbola and its conjugate.

7. Draw a parabola approximately; also a tangent parallel to a given chord.

8. Given an ellipse, the curve only, construct the center, axes, foci, directrices, and the tangent at some general point.

9. Like Ex. 8 for an hyperbola.

10. Given a parabola, construct the axis, focus, directrix, and the tangent at some general point. Why must the tangent be drawn in a different way than for the ellipse and hyperbola?

11. Using (42)–(44) as formulas write the equation of the diameter bisecting the chords of slope 2 in the curves $y^2=15\,x$, $4\,x^2+9\,y^2=36$, $x^2-y^2=16$, $x^2+y^2=25$. Illustrate by rough drawings.

12. Like Ex. 11 for chords of slope $\frac{1}{2}$ in the curves $y^2=x$, $9\,x^2+25\,y^2=900$, $4\,x^2-9\,y^2=36$, $x^2+y^2=9$.

13. Find the locus of the midpoints of chords drawn from the vertex of $y^2=12\,x$ to all points of the curve. Solve by using

(*a*) Rectangular coördinates, (*b*) Polar coördinates.

14. (*a*), (*b*). Like Ex. 13 (*a*), (*b*), for chords drawn from the left-hand vertex of $(x-4)^2+16\,y^2=16$.

15. Show by slopes that any diameter of a circle is perpendicular to the chords which it bisects.

16. In what sense could a diameter of a parabola be regarded as a line through the center? (Cf. § 247.)

§ 258. Contact Equation of a Tangent. The equation of the tangent line, at any given point (x_1, y_1) of a conic, can be written at sight by using a simple rule, proved in § 259 below.

To illustrate, consider the hyperbola

$$3x^2-8y^2+6x+7y-15=0. \tag{49}$$

Rewrite the equation in the form

$$3xx-8yy+3(x+x)+\tfrac{7}{2}(y+y)-15=0,$$

so that the variable part of every term is the product or sum of two x's, or two y's. Now *replace one of the x's everywhere by x_1, and one of the y's by y_1;* the result will be the equation of the tangent at (x_1, y_1):

$$3x_1x-8y_1y+3(x+x_1)+\tfrac{7}{2}(y+y_1)-15=0. \tag{50}$$

The same rule will work for any conic, with this addition: An xy term is rewritten $\frac{1}{2}(xy+xy)$; and x is replaced by x_1 in one term, and y by y_1 in the other.

Ex. I. Find the equation of the line tangent to the circle $x^2+y^2=25$ at (3, 4).
Rewritten: $xx+yy=25$.
Tangent: $3x+4y=25$.
Partial Check: (3, 4) lies both on this line and on the given circle.

Ex. II. Find the tangent to $xy=20$ at (4, 5).
Rewritten: $\frac{1}{2}(xy+xy)=20$.
Tangent: $4y+5x=40$. [Check?]

§ 259. Proof of the Rule. Consider the conic

$$ax^2+bxy+cy^2+dx+ey+f=0. \tag{51}$$

The slope at the point of tangency (x_1, y_1) is:

$$l=\frac{dy}{dx}=-\frac{2ax_1+by_1+d}{bx_1+2cy_1+e}. \tag{52}$$

Using this in the equation $y-y_1=l(x-x_1)$, and simplifying:

$$ax_1x+\frac{b}{2}(x_1y+xy_1)+cy_1y+\frac{d}{2}x+\frac{e}{2}y$$
$$=ax_1^2+bx_1y_1+cy_1^2+\frac{d}{2}x_1+\frac{e}{2}y_1. \qquad (53)$$

But, as (x_1, y_1) lies on the curve, equation (51) shows that the right member of (53) is equal to $\left(-f-\frac{d}{2}x_1-\frac{e}{2}y_1\right)$.

Substituting this in (53) and transposing, we obtain:

$$ax_1x+\frac{b}{2}(x_1y+xy_1)+cy_1y+\frac{d}{2}(x+x_1)+\frac{e}{2}(y+y_1)+f=0.$$

But this is precisely the result given by the rule of § 258, if we write equation (51) in the form

$$axx+\frac{b}{2}(xy+xy)+cyy+\frac{d}{2}(x+x)+\frac{e}{2}(y+y)+f=0, \qquad (54)$$

and substitute (x_1, y_1) as specified. So the rule holds.

For ready reference several standard contact equations are listed here. None need be memorized if we grasp clearly the idea of the rule.

Curve	*Tangent at* (x_1, y_1)	
$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1,$	$\frac{x_1x}{a^2}\pm\frac{y_1y}{b^2}=1.$	(55)
$y^2=4\,px,$	$y_1y=2\,p(x+x_1).$	
$xy=k^2,$	$x_1y+y_1x=2\,k^2.$	

§ 260. **Focal Radii and Tangents.** A line from a focus to any point on a conic is called a *focal radius.*

THEOREM. The tangent to an hyperbola at any point P bisects the angle between the focal radii FP and $F'P$.

Proof. The slopes of FP, TP, and $F'P$ are, respectively:

$$\frac{y}{x-c}, \qquad \frac{b^2x}{a^2y}, \qquad \frac{y}{x+c}.$$

The angle K which FP makes with TP is given by

$$\tan K = \frac{\dfrac{y}{x-c} - \dfrac{b^2x}{a^2y}}{1+\dfrac{y}{x-c}\dfrac{b^2x}{a^2y}} = \frac{a^2y^2-b^2x^2+b^2cx}{y[-a^2c+(a^2+b^2)x]}.$$

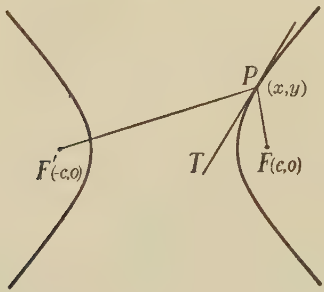


FIG. 152.

But, from the equation of the hyperbola, $a^2y^2-b^2x^2=-a^2b^2$. Also, $a^2+b^2=c^2$. Substituting:

$$\tan K = \frac{-a^2b^2+b^2cx}{y(-a^2c+c^2x)} = \frac{b^2}{cy}. \tag{56}$$

Likewise for $\angle K'$ which TP makes with $F'P$ we find:

$$\tan K' = \frac{b^2cx+a^2b^2}{y(c^2x+a^2c)} = \frac{b^2}{cy}. \tag{57}$$

Hence $K=K'$. (Q. E. D.)

For an ellipse a like proof shows that the same theorem holds, except that one radius should be extended. For a *parabola* the tangent bisects the angle between the focal radius FP and a line QP parallel to the axis. (Cf. § 257.)

These principles provide another easy method of drawing a tangent to an ellipse, parabola, or hyperbola, at any given point of the curve.

They also underlie the use of reflectors having the shape of a conic. A ray of light is always reflected in such a way that the angle between the directions of the ray before and after reflection is bisected by a line

tangent to the reflecting surface. From Fig. 153 we see, then, that rays of light emanating from one focus of an elliptic mirror must meet again at the other focus. (See also Ex. 11–12 below.)

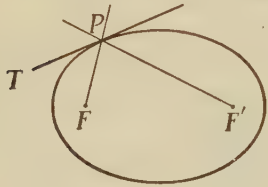


FIG. 153 *a*.

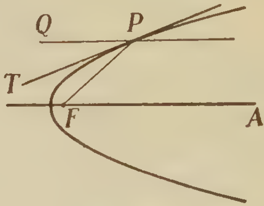


FIG. 153 *b*.

EXERCISES

1. Write the equation of the tangent to each following curve at the specified point. Draw an approximate figure.

(*a*) $4x^2+y^2=400$, $(6, -16)$; (*b*) $xy=50$, $(5, 10)$;
(*c*) $9x^2-16y^2=1296$, $(-20, 12)$; (*d*) $x^2=20y$, $(10, 5)$.

2. Like Ex. 1 for $x^2+2xy+y^2-10x+10y=0$, and $(0, 0)$.

3. Find the equation of the tangent to $A(x+h)^2+B(y+k)^2=C$ at (x_1, y_1) by multiplying out and using the foregoing rule. Show that the same result could be obtained by substituting immediately in $A(x+h)(x+h)+B(y+k)(y+k)=C$.

4. Write the equation of the tangent to $16(x+2)^2+25(y+1)^2=10000$ at $(13, 15)$. Draw a figure.

5. Like Ex. 4 for $(y-5)^2=12(x-7)$, at $(10, 11)$.

6. Test the homogeneity of equations (56)–(57).

7. (*a*) Verify equation (57). (*b*) Prove the theorem of § 260, for an ellipse. (*c*) Give a like proof in the case of a parabola.

8. Draw an ellipse carefully, and construct a tangent by using the principle of § 260. Check by another method.

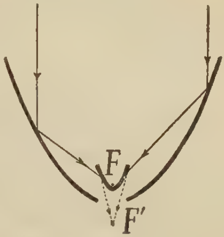
9. (*a*), (*b*). Like Ex. 8 for an hyperbola; also for a parabola.

10. In what sense could we say that, in all three kinds of conics, a tangent bisects an angle formed by two focal radii?

11. If rays of light emanate from the focus of a parabolic mirror, what must be their course after reflection?

12. Show that rays of light converging toward the focus of a (convex) hyperbolic mirror (from outside) will, after reflection, come together at the other focus.

[This principle is used in some telescopes, a small hyperbolic mirror being placed inside, and confocal with, a large parabolic reflector which has a small opening around the vertex. The eyepiece is placed at the other focus F', as in the adjacent figure.]



13. Find the distance of $(-2, 7)$ from the line tangent to $y^2 = 20\,x$ at $(5, 10)$.

14. Find the distance of each focus of $9\,x^2 + 25\,y^2 = 5625$ from the line tangent to this ellipse at $(-20, 9)$.

§ 261. The Foco-Symmetric Locus. About the left-hand focus F' of an hyperbola as center, let a circle be drawn with radius $2\,a$. And let the focal radii be drawn to any point P of the right-hand branch of the hyperbola, with $F'P$ meeting the circle at a point S. (Fig. 154.)

Then, by the original definition of the hyperbola, the difference of $F'P$ and FP is $2\,a$. Thus we have

$$FP = F'P - 2\,a, \qquad SP = F'P - 2\,a. \tag{58}$$

Hence P is equidistant from F and S; and as the tangent at P bisects $\angle FPS$, the points S and F must be symmetrically located with respect to this tangent.

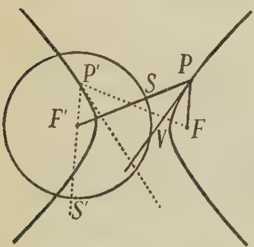


Fig. 154.

Thus, for every possible tangent to the hyperbola, the point S (or S') symmetric to F lies on the circle. And every point on the circle is obtained for some point P (or P'), if we include "points at infinity" where the tangent becomes an asymptote. Moreover, the entire figure could be reversed, right to left. Hence the theorem:

The locus of a point symmetric to either focus of an hyperbola, with respect to any tangent, is a circle of radius $2\,a$ about the other focus as center.

This locus we shall call the "foco-symmetric locus."

For an ellipse the same theorem holds. It may be proved similarly. For a parabola, the foco-symmetric locus is the *directrix*. (See Ex. 3, p. 419.)

§ 262. **Tangents from External Points.** The problem of drawing a tangent line to a conic, from a given external point P, is easily solved by using the foco-symmetric locus L for either focus. The points S, S', symmetric to F with respect to the tangents through P, lie at the same distance from P as does F. We can locate them by describing an arc with P as center and PF as radius, and cutting the foco-symmetric locus. Lines from P bisecting the lines FS and FS' are the required tangents.

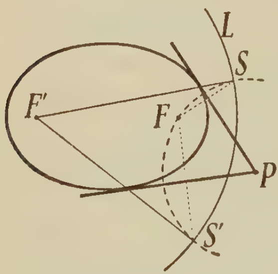


FIG. 155.

As a check observe that the points of tangency should lie on lines joining S and S' to the other focus F'.

§ 263. **Chord of Contact.** The equation of the line joining the points of tangency, of the two tangents to a conic from any given external point, is easily found.

Consider first the ellipse. And let (h, k) be the given external point; and (x_1, y_1) the point of contact of either tangent. The slope of the tangent is $-b^2x_1/a^2y_1$; hence

$$\frac{k-y_1}{h-x_1}=-\frac{b^2x_1}{a^2y_1}, \tag{59}$$

or $$b^2x_1h+a^2y_1k=b^2x_1^2+a^2y_1^2.$$

$$\therefore \quad \frac{hx_1}{a^2}+\frac{ky_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1. \tag{60}$$

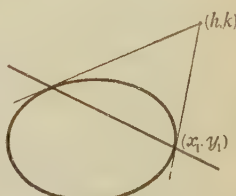


FIG. 156.

Thus the coördinates x_1, y_1 satisfy the *linear equation*

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1. \tag{61}$$

But (x_1, y_1) is *either* point of contact. Hence *both* points of contact lie on line (61); and (61) must be the required line.

This equation has the same form as the equation of the tangent when (h, k) is on the ellipse. A like result holds for the other conics. Hence the equation of the "chord of contact" can be written for any conic by the same rule as the contact equation of a tangent. (§ 258.)

EXERCISES

1. In Fig. 154 show that S' and F are symmetric with respect to the tangent at P'.

2. Prove the theorem of § 261 in the case of the ellipse.

3. Prove in two ways that the foco-symmetric locus in the case of the parabola is the directrix:

(*a*) By using the definition of the parabola, with Fig. 153 *b*;

(*b*) By regarding the parabola as the limiting form of the hyperbola in Fig. 154 as F' recedes to infinity, while F and V are fixed. [Show that the distance from V to the circle constantly equals VF.]

4. Draw an ellipse carefully and construct the two tangents to it from some (general) external point. Use the foco-symmetric locus for either focus. Check by the other. Also apply the check in § 262.

5. Like Ex. 4 for an hyperbola, drawn with fair accuracy.

6. Draw a parabola and construct the two tangents from some chosen (general) external point. What check should replace that mentioned in § 262 for the ellipse?

7. Derive the equation of the chord of contact for the tangents from an external point (h, k) to each following curve:

(*a*) $b^2x^2-a^2y^2=a^2b^2$, (*b*) $y^2=4\,px$.

8. Write by inspection the equation of the chord of contact for each following curve and external point:

(*a*) $x^2+y^2=100$, $(-50, 50)$; (*b*) $x^2-y^2=16$, $(2, -2)$;
(*c*) $x^2+4\,y^2=100$, $(14, 1)$; (*d*) $y^2=12\,x$, $(-1, 2)$.

9. (*a*)–(*d*). In Ex. 8 (*a*)–(*d*), find the intersections of each chord with the curve. Then write the contact equations of the tangents; and verify that each tangent passes through the given external point.

10. Find the equations of the tangents from (25, 0) to each following curve. Draw figures.

(*a*) $25\,x^2+16\,y^2=10{,}000$; (*b*) $xy=85$.

11. Find the equations of two tangents to $x^2-4\,y^2=64$, which have the slope $\frac{5}{8}$. [Hint: Find at what points on the curve the slope is $\frac{5}{8}$; then write each contact equation.]

12. Proceeding as in Ex. 11 show that the tangents of any slope l to the several conics have the following equations:

(*a*) Ellipse $b^2x^2+a^2y^2=a^2b^2$: $y=lx\pm\sqrt{l^2a^2+b^2}$;
(*b*) Hyperbola $b^2x^2-a^2y^2=a^2b^2$: $y=lx\pm\sqrt{l^2a^2-b^2}$;
(*c*) Parabola $y^2=4\,px$: $y=lx+p/l$.

§ 264. **Further Properties.** The conics have a vast number of geometric properties. We can mention only a few here.

(I) Theorem. The perpendiculars from the two foci of an ellipse to any tangent have a constant product.

Proof. The tangent at any point (x_1, y_1) is $b^2x_1x+a^2y_1y_1-a^2b^2=0$. Dividing by $\sqrt{b^4x_1{}^2+a^4y_1{}^2}$ and substituting either $(c, 0)$ or $(-c, 0)$ will give the length of one of the perpendiculars in question, — or else its negative. (§ 214.) Thus

$$p=\pm\frac{b^2x_1(c)-a^2b^2}{\sqrt{b^4x_1{}^2+a^4y_1{}^2}},\qquad p'=\pm\frac{b^2x_1(-c)-a^2b^2}{\sqrt{b^4x_1{}^2+a^4y_1{}^2}}. \tag{62}$$

Hence the product is

$$pp'=\frac{b^4(a^4-c^2x_1{}^2)}{b^4x_1{}^2+a^4y_1{}^2}. \tag{63}$$

By the equation of the ellipse, the denominator reduces to $a^4b^2-b^2x_1{}^2(a^2-b^2)$. And this becomes $b^2(a^4-x_1{}^2c^2)$, since $a^2-b^2=c^2$. Thus (63) gives $pp'=b^2$.

(II) Problem. Find the locus of the intersection of any two perpendicular tangents to an ellipse.

Experiment suggests a circle, concentric with the ellipse, and of radius $\sqrt{a^2+b^2}$.

If l be the slope of any one tangent, its equation [by Ex. 12 (*a*) above] is

$$y=lx\pm\sqrt{l^2a^2+b^2}. \tag{64}$$

The perpendicular tangent, of slope $-1/l$, is

$$y = -\frac{x}{l} \pm \sqrt{\frac{a^2}{l^2} + b^2}. \qquad (65)$$

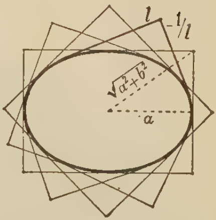


FIG. 157.

Rationalized and simplified, (64) and (65) become:

$$(y - lx)^2 = l^2a^2 + b^2, \qquad (66)$$

$$(ly + x)^2 = a^2 + l^2b^2. \qquad (67)$$

The intersection for any l can be found by solving (66) and (67) simultaneously. To find the rectangular equation of the locus, we must eliminate the parameter l. Simply adding (66) and (67), and canceling $(1 + l^2)$ throughout, gives

$$x^2 + y^2 = a^2 + b^2. \qquad (68)$$

The locus is the circle inferred experimentally.

EXERCISES

1. Prove the theorem of § 264 by using the equation of the tangent in the slope form (64).

2. Prove the corresponding theorem for the hyperbola.

3. Find the locus of the intersection of any two perpendicular tangents to the standard hyperbola.

4. Like Ex. 3 for the parabola $y^2 = 4\,px$.

5. Find the locus of the foot of a perpendicular drawn from a focus to any tangent to the ellipse.

6. (*a*) Like Ex. 5 for the hyperbola. (*b*) Also for the parabola.

7. Prove that any two tangents to a parabola meet on the diameter which bisects the chord of contact.

8. (*a*), (*b*). The same as Ex. 7 for the ellipse; also the hyperbola.

9. Prove geometrically that, if any ellipse and hyperbola have the same foci, they cross at right angles.

10. Aside from any possible degenerate case, what sort of conic is

(*a*) $x^2 + y^2 - xy + 16\,y = 0$? (*b*) $x^2 + 10\,xy - 8\,x = 15$?

11. Derive the equation of an ellipse with foci $(6, 5)$ and $(-2, -3)$, and with the longest diameter $2\,a = 12$. Also reduce your final equation to a standard form, and check.

CHAPTER XI

CURVATURE AND MOTION

PART I. CURVATURE AND EVOLUTES

§ 265. The Idea of Curvature. It is often important to know how fast the direction of a curve changes, per unit of length along the curve.

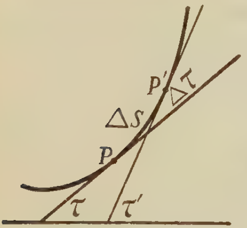


FIG. 158.

If the curve or its tangent line turns by 40° in 10 units of length, we say that it has an "average curvature" of 4° per unit. More generally, the average curvature of any arc, of length Δs, is $\Delta\tau/\Delta s$. (Fig. 158.)

The curvature K at a point P is defined as the limit of the average curvature of an arc including P, as the arc is indefinitely shortened:

$$K = \underset{\Delta s \to 0}{\mathrm{L}} \frac{\Delta \tau}{\Delta s} = \frac{d\tau}{ds}. \tag{1}$$

In pure mathematics radians are used instead of degrees, to simplify the formulas involved.

§ 266. Curvature of a Circle. In a circle, $\Delta\tau$ equals the central angle $\Delta\theta$ between the radii drawn to P and P'. Also $\Delta s = R\Delta\theta$, if R is the radius and θ is in radians.

$$\therefore \qquad \frac{\Delta\tau}{\Delta s} = \frac{\Delta\theta}{R\Delta\theta} = \frac{1}{R}. \tag{2}$$

I.e., the curvature of a circle, in radians per unit arc, is simply the reciprocal of the radius.

Ex. I. The curvature of a circle of radius 10 inches is .1 (radians per inch). The tangent turns by $.1^{(r)}$, or 5°.73, in each 1-inch arc.

§ 267. **General Formula.** In any plane curve $\tau=\tan^{-1} l$, where $l=dy/dx$. Hence by (33), p. 52,

$$\frac{d\tau}{ds}=\frac{\frac{dl}{ds}}{1+l^2}=\frac{\frac{dl}{dx}\cdot\frac{dx}{ds}}{1+l^2}. \tag{3}$$

But dl/dx is simply d^2y/dx^2; and by (66), p. 134, dx/ds is $1\div\sqrt{1+(dy/dx)^2}$.

$$\therefore \qquad \frac{d\tau}{ds}=\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}\cdot\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.$$

I.e.,

$$K=\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}. \tag{4}$$

This is the curvature in radians per unit length for any plane curve.

§ 268. **Circle of Curvature.** Any given curve G can be approximated closely by a circle for a short distance, — provided the circle is tangent to G and has its center on the concave side. Of all such tangent circles, the one which fits most closely is that which has *the same curvature as* G at the point of tangency P. (Fig. 159.) This latter circle C is called the "circle of curvature" for the curve G at P, its center the "center of curvature" and its radius the "radius of curvature."

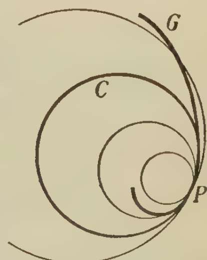


FIG. 159.

Since in a circle $R=1/K$, and K is the same for C as for G, the radius of curvature for any curve is:

$$R=\frac{1}{K}, \qquad \text{if } K\neq 0. \tag{5}$$

Hence, if we know the formula (4) for K, we can get R in any case by simply inverting the fraction in (4).

If K and R come out negative, — that is, if $d\tau/ds$ is negative, — this simply indicates that τ is decreasing as s (and x) increase; and shows that the curve G is *concave downward.*

In studying a curve near a point P, it is helpful to calculate R, lay it off on the normal, and draw the circle of curvature.

Ex. I. Find K and R for the cubical parabola $y=\frac{1}{3}x^3$.

Here $$\frac{dy}{dx}=x^2, \qquad \frac{d^2y}{dx^2}=2x.$$

$$\therefore \qquad K=\frac{2x}{(1+x^4)^{\frac{3}{2}}}, \qquad R=\frac{(1+x^4)^{\frac{3}{2}}}{2x}. \tag{6}$$

E.g., at $x=2$, $K=4/\sqrt{17^3}=.057$. The tangent is turning at the rate of .057[7] per unit; and $R=17.5$ units.

Again, at $x=0$, we find $K=0$ and $R=\infty$. At this point the tangent has instantaneously stopped turning; the "circle" of curvature has an infinite radius, and has become a straight line. (See Fig. 26 A, p. 62, for the shape of this curve near $x=0$.)

§ 269. K and R in Parametric Equations. If a curve is given by parametric equations, $x=f_1(t)$, $y=f_2(t)$, we find K and R by the same formulas (4) and (5) above. But we must note carefully that dy/dx and d^2y/dx^2 are derivatives to be taken *with respect to x* and not t.

Ex. I. Find R at the point where $t=2$ in the curve:

$$x=6t-t^3, \qquad y=1+t^2. \tag{7}$$

Differentiating as in Ex. I, p. 42:

$$\frac{dy}{dx}=\frac{2t}{6-3t^2}, \tag{8}$$

$$\frac{d^2y}{dx^2}=\frac{12+6t^2}{(6-3t^2)^3}. \tag{9}$$

When $t=2$, we find $dy/dx=-\frac{2}{3}$ and $d^2y/dx^2=-\frac{1}{6}$.

$$\therefore \qquad R=(1+\tfrac{4}{9})^{\frac{3}{2}}/(-\tfrac{1}{6})=-10.416.$$

The radius is 10.416 units; and the curve is concave downward.

EXERCISES

1. Plot $y=x^2$ from $x=-3$ to $x=3$, using the same scale for y as for x. Estimate the location of the center of curvature, for the lowest point; and measure the apparent R. Check by calculation.

2. In Ex. 1 draw tangents to the curve at $x=-.2$ and $x=.2$, and measure the change of direction. From this estimate the curvature at $x=0$. Check by calculation.

3. Plot $y=\sin x$, taking $x=0$, $\pi/6$, etc., to $x=\pi$, and using equal scales. Draw the apparent circle of curvature, for $x=\pi/2$; and check R by calculation.

4. Draw $9x^2+25y^2=900$ by inspection; also the apparent circle of curvature for one end of each axis. Calculate each R and check.

5. Calculate K and R for each following curve at the given point:

(*a*) $y=x^3-12x+10$, $(3, 1)$; (*b*) $y^2=x^3$, $(1, 1)$;
(*c*) $xy-50=0$, $(10, 5)$; (*d*) $y^2=6x$, $(6, -6)$;
(*e*) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, $(0, a)$; (*f*) $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$, $(a, 0)$.

6. Find R at one end of the latus rectum of

(*a*) $9x^2-16y^2=144$, (*b*) $y^2=12x$.

7. Find expressions for K and R at any point (x_1, y_1) of each following curve:

(*a*) $y=x^4$, (*b*) $y=e^x$,

(*c*) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, (*d*) $\left(\dfrac{x}{a}\right)^{\frac{2}{3}}+\left(\dfrac{y}{b}\right)^{\frac{2}{3}}=1$.

8. Verify equations (8), (9), of § 269.

9. (*a*) Find K and R for the curve $x=3t^2$, $y=3t-t^3$, at $t=1$. (*b*) Plot the curve from $t=-3$ to $t=3$ and compare.

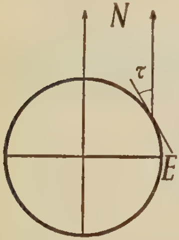
10. Find K and R at $\phi=\pi/4$ in each following curve:

(*a*) Ellipse,	$x=a\cos\phi$,	$y=b\sin\phi$;
(*b*) Hypocycloid,	$x=a\cos^3\phi$,	$y=a\sin^3\phi$;
(*c*) "Involute,"	$x=\cos\phi+\phi\sin\phi$,	$y=\sin\phi-\phi\cos\phi$;
(*d*) Epicycloid,	$x=a[4\cos\phi-\cos 4\phi]$,	$y=a[4\sin\phi-\sin 4\phi]$.

11. Show that the radius of curvature at any point of the cycloid, $x=a(\phi-\sin\phi)$, $y=a(1-\cos\phi)$, is $R=-2a\sqrt{2(1-\cos\phi)}$. [Why $-$?] What is R at the beginning of an arch? At the top? Draw roughly.

12. Find R for the hyperbola, $x=a\cosh u$, $y=a\sinh u$, at any point.

13. The curve of a certain steel beam 200 in. long, loaded uniformly, has the equation



$$y=-2\times10^{-9}(x^4-300\,x^3+4\,000\,000\,x).$$

Find R at $x=100$.

14. Like Ex. 13 if

$$y=-4\times10^{-12}(x^5-80\,000\,x^3+1\,600\,000\,000\,x).$$

15. Going north on the earth from the equator E, the elevation of the North Star N ($\angle\tau$ in the adjacent figure) increases less and less rapidly per mile. How does this indicate that there is flattening at the pole?

§ 270. Center of Curvature. The position of the center of curvature $C\ (X,\ Y)$ for any point $P\ (x,\ y)$ on a given curve G is readily calculated. By Fig. 160 we have, if τ is acute:

$$x-X=R\sin\tau,\qquad Y-y=R\cos\tau. \tag{10}$$

By (36), p. 378,

$$\sin\tau=\frac{\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}},$$

$$\cos\tau=\frac{1}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}.$$

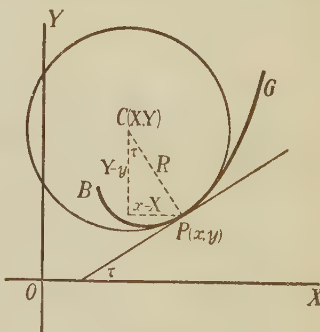


FIG. 160.

Using these values, with R as given by (5):

$$X=x-\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]\dfrac{dy}{dx}}{\dfrac{d^2y}{dx^2}},\qquad Y=y+\frac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}. \tag{11}$$

When τ is obtuse, the sign of $\cos\tau$ is to be changed above, but equations (11) are unaltered.

Ex. I. Find C for any point on the parabola

$$y^2 = 4\,px. \tag{12}$$

Differentiating implicitly:

$$\frac{dy}{dx} = \frac{2\,p}{y}, \tag{13}$$

$$\frac{d^2y}{dx^2} = -\frac{2\,p}{y^2}\frac{dy}{dx} = -\frac{4\,p^2}{y^3}. \tag{14}$$

Substituting in (11), simplifying and using (12):

$$\left.\begin{aligned} X &= x + \frac{y^2+4\,p^2}{2\,p} = x + \frac{y^2}{2\,p} + 2\,p = 3\,x + 2\,p, \\ Y &= y - \frac{(y^2+4\,p^2)y}{4\,p^2} = y - \frac{y^3}{4\,p^2} - y = -\frac{y^3}{4\,p^2}. \end{aligned}\right\} \tag{15}$$

Remark. These calculations could be shortened a little by first solving (12) for y (viz. $y = 2\sqrt{p}x^{\frac{1}{2}}$) and thus using explicit functions throughout. But that method is not possible in many cases.

§ 271. Railway Curves. To balance the centrifugal force as a railroad train rounds a curve, the outer rail must be higher than the inner. The amount of elevation needed depends upon the speed and curvature.

[For a speed of V mi./hr. and a curvature of $D°$ per unit length (which in railroad work is 100 ft.), the formula for the proper elevation (E in.) is

$$E = .00069\,DV^2. \tag{16}$$

E.g., on a 5° curve, for a speed of 40 mi./hr., $E = .00069(5)(40^2) = 5.52$. The outer rail should be about $5\frac{1}{2}$ in. higher than the inner. (See Ex. 13, p. 449.)]

Immediate transition from a straight track, level across, to a circular arc with steep banking, is impossible. The outer rail must rise gradually to the required height, and the curvature meanwhile increase correspondingly, from zero to the maximum degree desired. That degree can then be maintained in a circular arc as far as desired. But at the other end, transition to a straight track must be made by another tapering curve, of gradually diminishing curvature. Such curves are called "easement curves" or "spirals."

In a standard spiral the curvature increases at some constant rate ($C°$ per unit, gained per unit). And, as we shall see below, the parametric equations of the curve are:

$$\begin{aligned} x &= s(1-.000\,0076\,C^2s^4+\cdots), \\ y &= Cs^3\,(.002\,9089-.000\,000\,0158\,C^2s^4+\cdots), \end{aligned} \tag{17}$$

where s is the length of the spiral from its beginning to any point (x, y), and all distances are in units of 100 feet. For any chosen C, and for values of s as close together as desired, points (x, y) along the spiral can be calculated from (17). Tables of such points have been published.

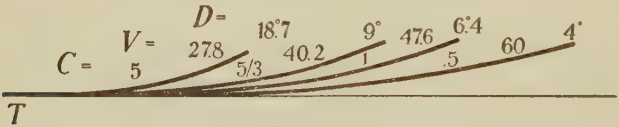


FIG. 161.

Fig. 161 shows several standard spirals, with the value of C for each, the greatest speed to which it is adapted, and the curvature at its end. This last is the greatest curvature suitable for the speed shown.

The method by which the equations (17) are derived is as follows. The curvature K increases at a constant rate, which in radian measure equals $\pi/180$ times C. Call this c. Then, at any point, $K = cs$; or by (1), p. 422, $d\tau/ds = cs$.

Integrating this:

$$\tau = \tfrac{1}{2} cs^2. \tag{18}$$

By (35), p. 378, we have for any curve whatever: $dx = \cos \tau \, ds$, $dy = \sin \tau \, ds$. With (18) these give

$$dx = \cos\left(\tfrac{1}{2} cs^2\right) ds, \qquad dy = \sin\left(\tfrac{1}{2} cs^2\right) ds. \tag{19}$$

Using the Maclaurin series for the cosine and sine, and integrating as was done in Ex. 12, p. 251, gives

$$x = s - \tfrac{1}{40} c^2 s^5 \cdots, \qquad y = \tfrac{1}{6} cs^3 - \tfrac{1}{336} c^7 s - \cdots. \tag{20}$$

Putting $c = (\pi/180)\, C = .017453\, C$, (20) reduce to (17).

Remarks. (I) The curvature of a circular railway curve is commonly defined as the change in direction for a unit *chord*, rather than a unit arc. Thus, in a "6° curve," a chord 100 ft. long subtends a central angle of 6°. The mathematical curvature would be about 5°.997. The slight adjustments are easily made when necessary. As a matter of fact, a so-called "approximation formula" given in some engineering handbooks amounts to using the strictly mathematical curvature.

(II) Various substitutes for the true spiral are sometimes used; *e.g.*, the cubic parabola, $y = .00291\, Cx^3$, which the spiral approximates closely at first, — also a series of circular arcs, each of greater curvature than the preceding. The spiral gives the smoothest results.

EXERCISES

1. Find the radius of a "3° curve" on a railway: (*a*) Using the mathematical definition of K; (*b*) Using the definition in Remark I.

2. (*a*), (*b*). Conversely, find the degree of a curve of radius 2000 ft., using each definition in Ex. 1 (*a*), (*b*).

3. From (16) calculate E for a 4° curve and 50 mi./hr. Likewise for an 8° curve and 30 mi./hr.

4. If a railroad limits E to 10 in., show that the greatest D allowable for any speed is $D = 14490/V^2$. What would be the largest permissible D for 60 mi./hr.? For 20 mi./hr.?

5. In the first spiral of Fig. 161 [$C = 5$], find x and y, and also D, at the point where $s = 2$. What is the length of the curve, in feet, from the beginning to the point where $D = 17.5$?

6. The greatest speed to which a given spiral is suited is found by the formula $V = 60/\sqrt[3]{2C}$. From this verify the values of V and D shown in Fig. 161 for the spiral $C = \frac{1}{2}$. Also find E in that spiral for the point where $s = 4$.

7. For a spiral $C = 4$, find as in Ex. 6 the greatest suitable V and D, also the total length of the curve, and E at the end.

8. Find the center of curvature of each following curve at the point specified; also draw freely.

(*a*) $y = x^2$, $(2, 4)$;
(*b*) $y = x^3$, $(1, 1)$;
(*c*) $y^2 = x^3$, $(1, -1)$;
(*d*) $xy = 20$, $(-2, -10)$;
(*e*) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, $(5, 0)$;
(*f*) $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$, $(3, 0)$;
(*g*) $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$, $(\frac{16}{5}, 3)$;
(*h*) $\dfrac{x^2}{64} - \dfrac{y^2}{36} = 1$, $(10, \frac{9}{2})$.

9. Like Ex. 8 for each following curve and point:

(*a*) $x = 20 \cos \phi$, $y = 12 \sin \phi$, where $\phi = \cos^{-1} \frac{4}{5}$;
(*b*) $x = 2 \cos^3 \phi$, $y = 2 \sin^3 \phi$, at $\phi = \pi/2$;
(*c*) $x = 10(\phi - \sin \phi)$, $y = 10(1 - \cos \phi)$, at $\phi = \pi/2$.

10. Find the center of curvature at any point (x_1, y_1) on:

(*a*) The ellipse, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$;
(*b*) The hyperbola, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

11. (*a*) Find the center of curvature C at any point $\phi = \phi_1$ on the curve $x = 2(\cos \phi + \phi \sin \phi)$, $y = 2 (\sin \phi - \phi \cos \phi)$.

(*b*) Show that C lies on a certain circle, whatever ϕ_1 may be.

12. Find the center of curvature for the spiral $C = 5$ in Fig. 161, at the point where $s = 2$.

13. Like Ex. 12 for the spiral $C = 1$, at $s = 5$.

§ **272. Evolute of a Curve.** The locus of the center of curvature C (Fig. 160, p. 426) as P travels along the given curve G is called the *evolute* of G.

Equations (11) give the position of C at any time in terms of x and y, or some parameter ϕ. To get the rectangular equation of the evolute, we must eliminate all variables other than X and Y.

Ex. I. Find the evolute of the parabola.

Experiment indicates a curve somewhat resembling the

semi-cubical parabola of Fig. 26, p. 62, but with its cusp to the right of the origin.

Solving equations (15) for x and y:

$$x = \tfrac{1}{3}(X - 2p) \qquad y = \sqrt[3]{-4p^2Y}.$$

Substituting these in the equation $y^2 = 4px$ gives

$$\sqrt[3]{16p^4Y^2} = \tfrac{4}{3}p(X - 2p),$$

$$\therefore \qquad Y^2 = \frac{4}{27p}(X - 2p)^3. \qquad (21)$$

This shows that the evolute actually is a semi-cubical parabola, — with its cusp at $(2p, 0)$. [How far from the focus?]

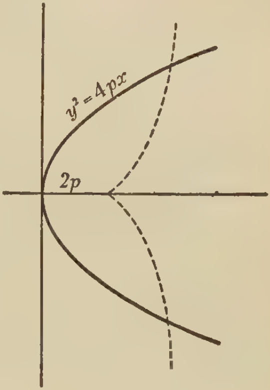


FIG. 162.

Ex. II. Find the evolute of the hypocycloid

$$x = a\cos^3\phi, \qquad y = a\sin^3\phi.$$

[Experiment is doubtful here without calculation.]

Differentiating as in § 29, we get on simplifying:

$$\frac{dy}{dx} = -\tan\phi, \qquad \frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\phi\sin\phi}. \qquad (22)$$

$$\begin{aligned} X &= x + 3a\sin^2\phi\cos\phi = a(\cos^3\phi + 3\cos\phi\sin^2\phi), \\ Y &= y + 3a\sin\phi\cos^2\phi = a(\sin^3\phi + 3\sin\phi\cos^2\phi). \end{aligned} \qquad (23)$$

To eliminate ϕ, observe that the terms in the right members of (23) are those which occur in the cube of a binomial, if combined.

$$\therefore \qquad X + Y = a(\cos\phi + \sin\phi)^3.$$

$$\text{Also} \qquad X - Y = a(\cos\phi - \sin\phi)^3.$$

Taking the cube root of each side, squaring, and adding, we get the required equation of the evolute:

$$(X + Y)^{\frac{2}{3}} + (X - Y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}. \qquad (24)$$

The $(X+Y)$ and $(X-Y)$ suggest a 45° rotation of axes. (§ 221.)

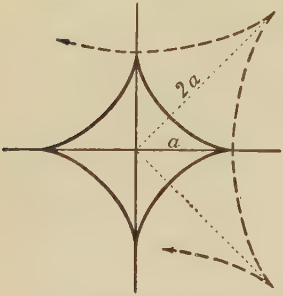


FIG. 163.

This gives

$$(\sqrt{2}\,x)^{\frac{2}{3}}+(-\sqrt{2}\,y)^{\frac{2}{3}}=2\,a^{\frac{2}{3}}, \qquad (25)$$

i.e.,
$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=(2\,a)^{\frac{2}{3}}. \qquad (26)$$

Hence the evolute is another hypocycloid, with axes twice as long as in the original curve and turned 45°.

Remark. It is often impossible to eliminate a parameter ϕ, or the original variables x and y. We then regard formulas (11), reduced, as *parametric equations* of the evolute.

EXERCISES

1. What must be the evolute of the larger hypocycloid found in Ex. II above? Answer without calculation. Draw roughly.

2. Find parametric equations and also the rectangular equation for the evolute of each following curve. Also, when the curve is familiar, try to see the shape of its evolute by inspection or experiment.

(*a*) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$; (*b*) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$;

(*c*) $x=3\,t^2, y=3\,t-t^3$; (*d*) $x=10/t, y=5\,t$;

(*e*) $x=10\cos t, y=10\sin t$; (*f*) $x=\cosh u, y=\sinh u$;

(*g*) $x=a\,(\cos\phi+\phi\sin\phi),\ y=a(\sin\phi-\phi\cos\phi)$.

3. What must the evolute of any circle be? Prove by calculation.

4. Show that the evolute of the ellipse in Ex. 2 (*a*) would coincide with a hypocycloid of four cusps, if all ordinates were reduced in the ratio $b:a$. What becomes of the evolute if $b=a$? [Note carefully the constants in that case.] Compare with Ex. 3.

§ 273. Relations. An important theorem is this:

Every normal to a given curve G is tangent to its evolute.

PROOF. For any point $P\,(x, y)$ of G, the center of curvature $C(X, Y)$ on the evolute is, by (10), p. 426:

$$X=x-R\sin\tau, \qquad Y=y+R\cos\tau. \qquad (27)$$

If we vary s (the length of the arc BP in Fig. 164), P will move along G, and C along E. Differentiating (27) with respect to s:

$$\frac{dX}{ds}=\frac{dx}{ds}-R\cos\tau\frac{d\tau}{ds}-\sin\tau\frac{dR}{ds},$$
$$\frac{dY}{ds}=\frac{dy}{ds}-R\sin\tau\frac{d\tau}{ds}+\cos\tau\frac{dR}{ds}. \qquad (28)$$

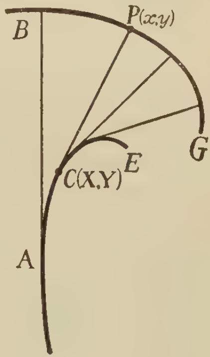


FIG. 164.

But $dx/ds=\cos\tau$, $dy/ds=\sin\tau$, and $d\tau/ds$ is the curvature K or $1/R$. Thus equations (28) reduce to

$$\frac{dX}{ds}=-\sin\tau\frac{dR}{ds},\qquad \frac{dY}{ds}=\cos\tau\frac{dR}{ds}. \qquad (29)$$

Dividing dY/ds by dX/ds gives

$$\frac{dY}{dX}=-\operatorname{ctn}\tau=-\frac{1}{dy/dx}.$$

Hence the tangent to the evolute at C has the same slope as the normal CP to curve G at P; and coincides with it.

THEOREM II. *The length of any arc of the evolute E is equal to the difference of the radii of curvature of G drawn from the ends of the arc.*

PROOF. Squaring (29) and adding:

$$\left(\frac{dX}{ds}\right)^2+\left(\frac{dY}{ds}\right)^2=\left(\frac{dR}{ds}\right)^2,\quad i.e.,\ \frac{dS}{ds}=\pm\frac{dR}{ds},$$

where S is the length of arc of the evolute E. Integrating:

$$S=\pm R+k. \qquad (30)$$

Thus the length of arc of E between two points must equal the difference of the radii of G at its ends.

§ 274. **Involutes.** Let an ideal thread, — *i.e.*, a flexible geometrical line, — be wound upon a curve E or unwound from it, the free end being held taut. The path G of any

point P on the thread is called an *involute* of the curve E. (Fig. 164.) A curve has any number of involutes, traced by different points P along the thread.

Any curve is the evolute of each of its various involutes. This may be proved by calculation or seen as follows: The conditions of motion for P are momentarily the same as if C were *fixed* and CP were revolving about C, — *i.e.*, as if P were traveling in a circle of radius CP with center C. Hence G is instantaneously bending with a radius of curvature CP and center of curvature C. Therefore E, the locus of C, is the evolute of G, which is any one of the involutes of E.

And, conversely, any given curve is an involute of its evolute.

For, in Fig. 164, suppose we start at A to wind upon E a thread of length L, equal to the radius of curvature of the given curve G at B. At any point C, the unwound part, being tangent to the evolute, will fall along a normal of G. (See Theorem I, § 273.) And by Theorem II, the length of that part will equal the radius of curvature of the given curve drawn from C. Hence the unwound part CP will always just reach the curve G, and P will describe the latter.

§ 275. The Cycloidal Pendulum. — These principles find an interesting application in the case of the cycloid. By Ex. 8, p. 435, the evolute of a cycloid is an equal cycloid, displaced as shown by the dotted curve in Fig. 165, when both are inverted. We could, therefore, construct a pendulum whose end (x, y) would travel along an inverted cycloid, by causing the cord to swing between cycloidal guides. Such a pendulum would swing with a period strictly independent of the amplitude, or maximum angle reached, — as is proved in treatises on mechanics. An ordinary pendulum does not quite do so, — although, for all very small amplitudes, the time is nearly constant.

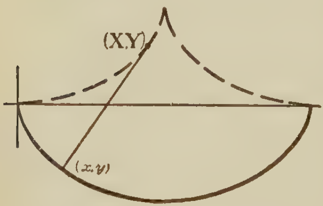


Fig. 165.

EXERCISES

1. If a straight line L be rolled along any curve C without slipping, what relation will the roulette traced by any point on L bear to C?

2. How could a pendulum bob be made to travel along the involute of an ellipse?

3. If a tethered horse grazes around a circular wall, gradually winding up the rope, what curve bounds the area covered?

4. Find the complete length of the evolute of an ellipse, of semi-axes 5 and 3, by merely using values of R for the ellipse.

5. Like Ex. 4 for the evolute of $y^2 = 20\,x$ from the cusp to the point corresponding to $x = 5$ on the parabola.

6. By *Intro.*, § 274, the involute of $x^2 + y^2 = a^2$ is $x = a(\cos\phi + \phi\sin\phi)$, $y = a(\sin\phi - \phi\cos\phi)$. Find the values of R for the latter at $\phi = 0$ and $\phi = \pi/2$; and compare with the length of the corresponding quarter-circle.

7. Find the rectangular equation of the evolute of each following curve; and draw a rough figure:

(a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, (b) $xy = 2\,a^2$.

[Hint: After finding X and Y, form $X + Y$ and $X - Y$.]

8. Show that the evolute of the common cycloid is $X = a(\phi + \sin\phi)$, $Y = a(-1 + \cos\phi)$. In these equations put $\phi = \phi' + \pi$; and show that the evolute is a cycloid equal to the original but displaced. Illustrate by a figure.

§ 276. Elastic Curve of a Beam. A loaded beam, supported only at its ends, sags slightly. The lower surface is stretched, the upper compressed. Somewhere between there is a neutral surface which retains its normal length. An axial line running lengthwise in the neutral surface is called the elastic curve of the beam, for the given loads and supports.

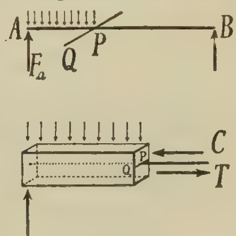


FIG. 166.

The equation of the elastic curve is found by integrating a simple differential equation; and the latter is derived once for all as follows:

Consider a portion of the beam x in. long, extending from the left end A to any point P. (Fig. 166.) The bending

moment at P, say M_x as defined in § 72, is the algebraic sum of the torques about P of all the external forces applied to the portion AP. Or, we may regard these torques as taken about a horizontal axis PQ perpendicular to the beam. For equilibrium this bending moment, M_x, must be balanced by the total torque about PQ of all the internal forces, compression C and tension T, exerted upon the portion AP by the portion PB of the beam to the right of P.

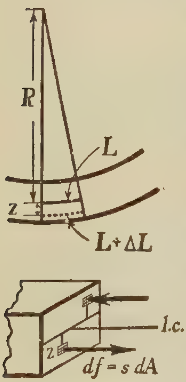


FIG. 167.

Both C and T are distributed forces; and their total torque is found by integration. The stress, s lb. per sq. in., varies with the distance z in. from PQ. (We shall here regard z as positive downward and negative upward.) In fact, s is proportional to the elongation ΔL of a lengthwise element, being

$$s = E\,\Delta L/L, \tag{31}$$

where E is the "modulus of elasticity," — a constant for any one material, — and L is the natural length of the element. Further, if R be the radius of curvature of the elastic curve at P, we have by Fig. 167,

$$\frac{L+\Delta L}{L} = \frac{R+z}{R}, \tag{32}$$

$$\therefore \qquad \frac{\Delta L}{L} = \frac{z}{R}. \tag{33}$$

Observe that when z is negative (33) gives a negative value for ΔL, which agrees with the fact that the elements above P are shortened.

Combining (31) and (33):

$$s = \frac{E}{R} z. \tag{34}$$

Hence the elementary force df acting upon any tiny area dA is $df = s\,dA = (E/R)z\,dA$; and its moment about PQ is

$$z\,df = \frac{E}{R} z^2\,dA. \tag{35}$$

The total for the entire distributed C and T, which must balance the bending moment, is then:

$$M = \frac{E}{R}\iint z^2\,dA = \frac{EI}{R}, \tag{36}$$

where I is the second moment of area of the cross section about PQ. (See p. 131.)

In the formula for R (p. 423), we may ignore the term $(dy/dx)^2$ for any ordinary loading of a beam, since the slope is very small at every point. Then (36) gives

$$EI\frac{d^2y}{dx^2} = M_x. \tag{37}$$

This is the general differential equation for any such beam. To use it we calculate M_x by proceeding as in § 72, *Remark*, and then changing X to x. Also, we find I for a cross section of the beam; use the value of E for the material in question; and integrate twice.

Ex. I. Find the elastic curve for the beam in Ex. I, § 72, if it is 1.5 in. wide, 6 in. high, and made of wood for which $E = 1\,500\,000$.

For a rectangular cross section, and an axis PQ through its middle, we find

$$I = \iint z^2\,dA = \int_{-3}^{3} z^2\,(1.5)\,dz = 27. \tag{38}$$

And, by (49), p. 123: $M_x = .002\,x^3 - 1.5\,x^2 + 130\,x$. Hence (37) becomes

$$\frac{d^2y}{dx^2} = \frac{1}{40\,500\,000}[.002\,x^3 - 1.5\,x^2 + 130\,x]. \tag{39}$$

Integrating twice:

$$y=\frac{1}{40\,500\,000}\left[.0001\,x^5-\tfrac{1}{8}\,x^4+\tfrac{65}{3}\,x^3\right]+c_1x+c_2.$$

Since $y=0$ at each end ($x=0$, $x=100$), we find

$$c_2=0, \qquad c_1=-\frac{1}{40\,500\,000}\left[\frac{610\,000}{6}\right]. \tag{40}$$

Whence finally

$$y=\frac{1}{40\,500\,000}\left[.0001\,x^5-\tfrac{1}{8}\,x^4+\tfrac{65}{3}\,x^3-\tfrac{610\,000}{6}\,x\right]. \tag{41}$$

§ 277. **Cantilever Beams.** If a beam is embedded or rigidly fixed at one end A and unsupported at the other, it is called a "cantilever beam." The reaction of the support at A is not obtainable in advance; so we calculate M_x by considering the external forces applied to the portion PB of the beam to the right of P, instead of that to the left.

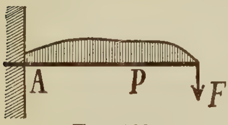


FIG. 168.

This is legitimate, provided we regard counterclockwise torques as positive; for each external set must be balanced by a set of internal forces acting on the section at P, in one direction or the other, — numerically equal.

To find the constants of integration we note that y and dy/dx are both zero at $x=0$. (Fig. 168.) But y is not zero at the other end as before.

If a beam embedded at A is supported or embedded at B, or if there is a concentrated load between the supports, still other modifications of procedure are necessary.*

EXERCISES

Each beam has a rectangular cross section. Take E = 30 000 000 for steel, or 1 500 000 for wood.

1. Find the equation of the elastic curve for a wooden beam resting on piers at its ends ($x=0$, $x=100$), if $I=80$ and $M_x=600\,x-6\,x^2$.

2. In Ex. 1 where does the minimum value of y or the maximum "deflection" occur? How large is it?

*See L. A. Martin: *Textbook of Mechanics*, v. 3.

3. Like Ex. 1 for a steel beam, if $I=\frac{10}{3}$, and $M_x=2600\,x-24\,x^2-.02\,x^3$.

4. Find the equation of the elastic curve for a steel beam 50 in. long, embedded at $x=0$, if $I=4$ and $M_x=1200\,(x-50)$. What is the deflection at $x=50$?

5. Find I for a wooden beam 2 in. wide and 8 in. high. If embedded at $x=0$, and so loaded that $M_x=512\,(x-60)$, find the equation of the elastic curve. Also find the deflection at $x=60$.

6. A steel beam 200 in. long, resting on piers at its ends, carries a load of 60 lb. per in., including its own weight. Show that $M_x=6000\,x-30\,x^2$. Also find the equation of its elastic curve, if it is $\frac{1}{2}$ in. wide and 12 in. high.

7. A wooden beam, 2 in. $\times$ 6 in., is embedded at one end ($x=0$), and carries only a load of 360 lb. at its free end ($x=50$), — ignoring its own weight. Show that $M_x=360(x-50)$. Also find the equation of the elastic curve, and the deflection at $x=50$.

8. Find I and M_x for beams having the following sections, supports and loads (w lb. per in.):

(a) 1 in. $\times$ 12 in., on piers ($x=0$, $x=80$), $w=50$;

(b) 2 in. $\times$ 6 in., on piers ($x=0$, $x=100$), $w=24-.24\,x$;

(c) 1 in. $\times$ 10 in., embedded ($x=0$), free ($x=60$), $w=6$;

(d) $\frac{1}{2}$ in. $\times$ 12 in., as in (c), $w=20+3\,x$.

9. (a) – (d). Find the equation of each elastic curve in Ex. 8, if beams (a) and (d) are steel; and (b) and (c) are wooden.

§ 278. K and R in Polar Coördinates.

If the equation of a curve is given in polar coördinates, the curvature can be found from the following formula, and R from the fact that $R=1/K$:

$$K=\frac{r^2+2\left(\dfrac{dr}{d\theta}\right)^2-r\dfrac{d^2r}{d\theta^2}}{\left[r^2+\left(\dfrac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}. \qquad (42)$$

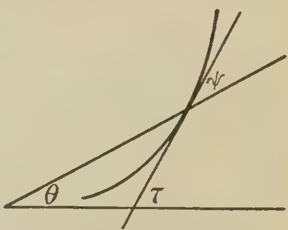


Fig. 169.

Proof. Since $\tau=\theta+\psi$,

$$\therefore \qquad K=\frac{d\tau}{ds}=\frac{d\theta}{ds}+\frac{d\psi}{ds}. \qquad (43)$$

In (46), p. 383, denote $dr/d\theta$ by r'. Then

$$\psi = \tan^{-1}\left(\frac{r}{r'}\right). \tag{44}$$

Differentiating with respect to s, by (38), p. 55:

$$\frac{d\psi}{ds} = \frac{d\psi}{d\theta}\cdot\frac{d\theta}{ds} = \frac{\frac{d}{d\theta}\left(\frac{r}{r'}\right)}{1+\frac{r^2}{r'^2}}\frac{d\theta}{ds}. \tag{45}$$

Differentiating the fraction r/r' by (12), p. 39, and then multiplying numerator and denominator of (45) by r'^2:

$$\frac{d\psi}{ds} = \frac{r'^2 - r\frac{dr'}{d\theta}}{r'^2+r^2}\frac{d\theta}{ds}. \tag{46}$$

Using this in (43) and factoring out $d\theta/ds$:

$$K = \frac{d\theta}{ds}\left[1+\frac{r'^2 - r\frac{dr'}{d\theta}}{r'^2+r^2}\right] = \frac{d\theta}{ds}\left[\frac{r^2+2\,r'^2 - r\frac{dr'}{d\theta}}{r'^2+r^2}\right]. \tag{47}$$

Replacing r' by $dr/d\theta$, and $dr'/d\theta$ by $d^2r/d\theta^2$, and recalling from (68), p. 135, that $ds/d\theta = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$, we obtain formula (42).

Remark. We shall not use (42) enough to justify memorizing it. Refer to it here as needed.

EXERCISES

1. Find K and R for the spiral $r = e^\theta$ at $\theta = 0, 1, 2$.

2. Like Ex. 1 for the spiral $r = 2\,\theta$.

3. Find by inspection the value of R at any point of each following curve. Also calculate R by (42) and compare.

(*a*) $r = 10\cos\theta$, (*b*) $r = 20\sin\theta$.

4. Find R at the outer end of a loop of each following curve. Draw the loop and circle of curvature roughly.

(a) $r=10 \sin 2\theta$, (b) $r=20 \sin 3\theta$,
(c) $r=5 \cos 2\theta$, (d) $r=2 \cos 3\theta$.

5. Find R at the point nearest the pole on each following conic. Illustrate roughly.

(a) $r=\dfrac{15}{1+\frac{1}{2}\cos\theta}$, (b) $r=\dfrac{15}{1-2\cos\theta}$,
(c) $r=\dfrac{15}{1-\cos\theta}$, (d) $r=10 \sec^2 \dfrac{\theta}{2}$.

6. Find R for the parabola $r=20/(1+\cos\theta)$ at the upper end of the latus rectum.

7. Find R at a general point of each following curve:

(a) Cardioid, $r=a(1-\cos\theta)$; (b) Lemniscate, $r^2=a^2 \cos 2\theta$;
(c) Conic, $r=p/(1-e\cos\theta)$, (d) Hyperbola, $r^2=a^2 \sec 2\theta$.

8. Find the equation of the elastic curve of a steel beam, $\frac{3}{2}$ in. $\times 6$ in., embedded at one end ($x=0$) and free at the other ($x=60$), if it carries a load as follows:

(a) Only its own weight, 2.5 lb. per in. length;
(b) Its own weight as in (a), and a 4000 lb. load at $x=60$;
(c) Its weight as in (a), plus w lb./in., where $w=.6\,x$.

9. Find K and R for the curve $y=e^{-x}$ at $x=0$.

10. A point moved thus: $x=10\,t$, $y=80\,t-16\,t^2$. Find the radius of curvature of its path at the highest point.

PART II. MOTION IN A PLANE

§ 279. General Plan. A few important types of motion have already been studied briefly. We proceed now to consider some general principles, which can be used to study any motion in a given plane.

The basic idea is this: As a point moves, its coördinates (x, y) or (r, θ) vary with the time t in some definite way. If we can obtain the *equations of motion*, — *i.e.* equations expressing x and y, or r and θ, as functions of t, — the motion will be definitely known.

Positions along the path can be calculated by substituting values for t; and various facts as to the speed and acceleration can be found by appropriate differentiations. The ordinary rectangular or polar equation of the path can be found by eliminating t.

The equations of motion are, in fact, simply parametric equations, with t as the parameter. They are often obtained by starting from some known fact as to the forces which produce the motion, expressing this fact by a differential equation, and integrating.

§ 280. **Component Speeds and Accelerations.** The rates at which x and y are increasing at any instant are called the "X- and Y-component speeds," v_x and v_y:

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}. \tag{48}$$

By *Intro.*, pp. 486–7, the actual speed and direction of motion are found from these components by the parallelogram method; or

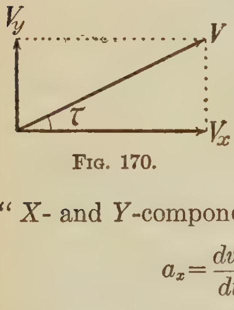


FIG. 170.

$$v = \sqrt{v_x^2 + v_y^2}, \qquad \tan \tau = \frac{v_y}{v_x}. \tag{49}$$

In like manner, the rates at which v_x and v_y are increasing are called the "X- and Y-component accelerations," a_x and a_y:

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}. \tag{50}$$

And these components, combined by the parallelogram method, give what is called the "total acceleration." (See §§ 281–82 below.) That is, the magnitude a and direction angle A of the acceleration are given by:

$$a = \sqrt{a_x^2 + a_y^2}, \qquad \tan A = \frac{a_y}{a_x}. \tag{51}$$

§ 281. Velocity, a Directed Quantity. To show how a point P is moving at any instant we draw a directed line or *vector*. Its length shows the speed and its direction the direction of motion. (Fig. 171 *a*.) If P moves along a curved path with a variable speed, the vector turns and also changes length. This varying vector represents a varying physical quantity called *velocity*.

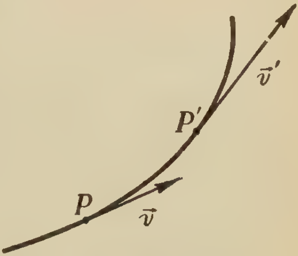


FIG. 171 *a*.

Speed is merely the rate of motion, such as 100 ft./sec. Velocity is *speed with a direction assigned to it.* We shall denote velocity by $\vec{v}$ (read "directed v"), to distinguish it from the speed v. Of course v is the numerical value or magnitude of $\vec{v}$.

As in the case of forces and other directed quantities, velocities are added and subtracted according to the parallelogram method. Thus, in Fig. 171 *b*:

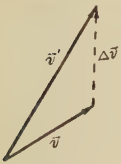


FIG. 171 *b*.

$$\vec{v} + \overrightarrow{\Delta v} = \vec{v}', \tag{52}$$

or

$$\overrightarrow{\Delta v} = \vec{v}' - \vec{v}. \tag{53}$$

The combined *effect* of velocities $\vec{v}$ and $\overrightarrow{\Delta v}$ for 1 sec. would be the same as the effect of $\vec{v}'$ for 1 sec. The sum of the numerical speeds, represented by the *lengths* of the vectors, would be different.

§ 282. Acceleration. Acceleration in general is defined as the rate at which the *velocity* is changing. For a straight path this is the same as the rate at which the *speed* is changing, but not for a curved path.

In Fig. 171 *b* let $\vec{v}$ be the velocity at any instant and $\vec{v}'$ the velocity Δt sec. later. The difference or change in velocity

is $\overrightarrow{v'} - \overrightarrow{v}$ or $\overrightarrow{\Delta v}$. If Δt is made much smaller, $\overrightarrow{v'}$ will be nearer $\overrightarrow{v}$, $\overrightarrow{\Delta v}$ will be much smaller, and its direction probably a little different. The ratio $\overrightarrow{\Delta v}/\Delta t$ is a vector quantity. It usually approaches a limit as $\Delta t \to 0$; and this limit, also a vector quantity, is the acceleration $\overrightarrow{a}$. The numerical value of $\overrightarrow{a}$, denoted by a, is the limit of the numerical value of the ratio $\overrightarrow{\Delta v}/\Delta t$; and the direction of $\overrightarrow{a}$ is the limiting direction of $\overrightarrow{\Delta v}$.

Formulas (51) for a and for the angle A which $\overrightarrow{a}$ makes with the positive X-axis are derived as follows.

The X-component of $\overrightarrow{\Delta v}$ (Fig. 171 b) equals the difference of the X-components of $\overrightarrow{v'}$ and $\overrightarrow{v}$; and hence is simply the change in v_x, that is, Δv_x. Likewise the vertical component of $\overrightarrow{\Delta v}$ is Δv_y. Hence the numerical value of $\overrightarrow{\Delta v}/\Delta t$ is

$$\text{Num. Val. } \frac{\overrightarrow{\Delta v}}{\Delta t} = \sqrt{\left(\frac{\Delta v_x}{\Delta t}\right)^2 + \left(\frac{\Delta v_y}{\Delta t}\right)^2}. \tag{54}$$

Taking limits as $\Delta t \to 0$ gives, as stated in (51):

$$a = \sqrt{a_x^2 + a_y^2}.$$

Calling the direction angle of $\overrightarrow{\Delta v}$ at any instant A':

$$\tan A' = \frac{\Delta v_y}{\Delta v_x} = \frac{\Delta v_y/\Delta t}{\Delta v_x/\Delta t}. \tag{55}$$

Taking limits as $\Delta t \to 0$, and remembering that A is by definition the limit of A', we get the second equation in (51), viz.

$$\tan A = \frac{a_y}{a_x}.$$

Remark. The physical law $F = ma$ applies to the true acceleration as defined above, not to the mere rate of change of speed. Force is needed to change the *direction*, even if the speed is constant.

Ex. I. A point moved thus: $x=40\,t^2$, $y=t^3$. Find its position, velocity, and acceleration at $t=10$.

At $t=10$ the position was $x=4000$, $y=1000$.

$$v_x=\frac{dx}{dt}=80\,t, \qquad v_y=\frac{dy}{dt}=3\,t^2.$$

Substituting $t=10$ gives $v_x=800$, $v_y=300$, whence

$$v=\sqrt{800^2+300^2}, \quad \tan\tau=\tfrac{3}{8}. \tag{56}$$

That is, the speed or magnitude of velocity was 854.4, and the direction angle was $\tau=\tan^{-1}\frac{3}{8}=21°$, approx.

$$a_x=\frac{d^2x}{dt^2}=80, \qquad a_y=\frac{d^2y}{dt^2}=6\,t,=60.$$

$$\therefore \qquad a=\sqrt{80^2+60^2}=100, \quad \tan A=\tfrac{60}{80}=.75. \tag{57}$$

The acceleration was 100 units, in a direction inclined about 37°. Thus the direction of acceleration made an angle of $37°-21°$, or 16°, with the direction of motion.

Remark. To get the equation of the path, observe that cubing x and squaring y will give t^6 in each case, with some coefficient. Thus

$$x^3=(40)^3y^2. \tag{58}$$

This is a semi-cubical parabola, $y^2=kx^3$.

EXERCISES

1. For each following motion find the position, velocity, and acceleration at $t=2$. Draw vectors representing v_x, v_y, $\overrightarrow{v}$, and a_x, a_y, $\overrightarrow{a}$.

(*a*) $x=2\,t^2+10$, $y=3\,t-t^3$; (*b*) $x=t^2-2\,t$, $y=\frac{8}{3}\,t^{\frac{3}{2}}$;

(*c*) $x=72-6\,t^2$, $y=12\,t-t^3$; (*d*) $x=10\cos t$, $y=10\sin t$.

2. In Ex. 1 (*d*) what is the path of the moving point?

3. A batted ball moved thus: $x=128\,t$, $y=4+160\,t-16\,t^2$. Find the position, velocity, and acceleration when highest; also at $t=2$. Represent by vectors.

4. A point traveled in an ellipse, of semi-axes 10 in. and 6 in., with the eccentric angle ϕ varying thus: $\phi=3\,t$. Find the position, velocity, and acceleration at $t=\pi/6$.

5. A circle of radius 5 in., rolling along a straight line without slipping, turned at the rate of $2^{(r)}$ per sec. Find the position, velocity, and acceleration of a point on the circle, when at the same height as the center, and rising.

6. A string was unwound from a circle of radius 20 in. so that the point of contact moved through $1^{(r)}$ per sec. Find the position, velocity, and acceleration of the free end of the string 2 sec. after starting.

7. In Ex. 3 find the curvature of the path at the highest point.

8. In Ex. 1 (a) find K and R for the path, at $t=2$.

9. In Ex. 6 find R at any point of the path.

§ **283. Tangential Acceleration.** The rate at which the *speed* changes is called the tangential acceleration:

$$a_t=\frac{dv}{dt}. \tag{59}$$

This also equals the component of the total acceleration $\overrightarrow{a}$ along the direction of motion. (Fig. 172.)

For, multiplying a by the cosine of the angle $(A-\tau)$ which $\overrightarrow{a}$ makes with the tangent line:

$$a \cos (A-\tau)=a (\cos A \cos \tau+\sin A \sin \tau).$$

But $a \cos A$ is the horizontal component of $\overrightarrow{a}$, or a_x; and $a \sin A=a_y$. Also $\cos \tau=v_x/v$, and $\sin \tau=v_y/v$.

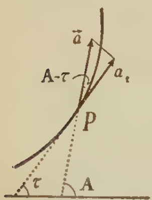


FIG. 172.

$$\therefore \qquad a \cos (A-\tau)=a_x\frac{v_x}{v}+a_y\frac{v_y}{v}$$

$$=\frac{v_x\dfrac{dv_x}{dt}+v_y\dfrac{dv_y}{dt}}{v}. \tag{60}$$

But a_t is the derivative of v or $\sqrt{v_x^2+v_y^2}$; and this derivative is precisely the right member of (60). Hence

$$a \cos (A-\tau)=a_t, \tag{61}$$

as stated.

§ **284. Normal Acceleration.** The component of $\overrightarrow{a}$ along the *normal*, that is, perpendicular to the direction of motion,

can be found by multiplying a by the cosine of the angle between $\vec{a}$ and the normal.

By a rather long calculation this product can be reduced to the formula

$$a_n = \frac{v^2}{R}, \tag{62}$$

where R is the radius of curvature. The formula can also be derived in another way. (§ 285.)

Formula (62) can be employed whether we are using rectangular or polar coördinates.

Observe that for a constant speed v, the curvature or rate of bending, $K = 1/R$, is directly proportional to the normal acceleration. Hence, if the latter is zero, $K = 0$; and the path is straight.

§ 285. Alternative Derivation of (62). In Fig. 173 the component of $\overrightarrow{\Delta v}$ along the normal PN must equal that of $\vec{v}'$. The angle between $\vec{v}'$ and $\vec{v}$ is $\Delta\tau$; hence that between $\vec{v}'$ and PN is $90° - \Delta\tau$, and the numerical value of the normal component of $\vec{v}'$ is

$$v' \cos (90° - \Delta\tau) = v' \sin \Delta\tau.$$

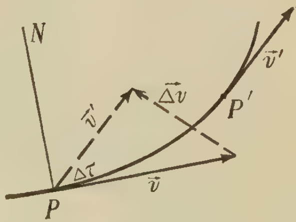


Fig. 173.

Using Maclaurin's series: $\sin x = x - x^3/3! + \cdots$, and replacing x by $\Delta\tau$, $v' \sin \Delta\tau$ becomes

$$v'\left(\Delta\tau - \frac{\Delta\tau^3}{3\,!} + \frac{\Delta\tau^5}{5\,!} - \cdots\right).$$

Hence the component of $\overrightarrow{\Delta v}/\Delta t$ along the normal is

$$v' \frac{\Delta\tau\left(1 - \frac{\Delta\tau^2}{3\,!} + \frac{\Delta\tau^4}{5\,!} \cdots\right)}{\Delta t}. \tag{63}$$

The limit of this as $\Delta t \to 0$ is the component of $\vec{a}$ along the normal, viz. a_n. But the limit of v' is v; and the limit of the fraction in (63) is $d\tau/dt$.

$$\therefore \qquad a_n = v\frac{d\tau}{dt}. \tag{64}$$

We do not know $d\tau/dt$ directly; but we know that $d\tau/ds$ is the curvature K, and we may write:

$$\frac{d\tau}{dt} = \frac{d\tau}{ds}\frac{ds}{dt} = Kv. \tag{65}$$

$$\therefore \qquad a_n = v(Kv) = \frac{v^2}{R}. \tag{66}$$

EXERCISES

1. Find the position, velocity, and acceleration, at $t = 3$, for a point moving thus: $x = t^2$, $y = t^3$. Also find a_t and a_n; and verify that $a_t^2 + a_n^2 = a^2$. Represent by vectors with the correct directions.

2. Like Ex. 1 for $x = 60\,t$, $y = 80\,t - 16\,t^2$.

3. Like Ex. 1 for $x = 10 \cos t$, $y = 10 \sin t$; at $t = \pi/4$.

4. If a point moves in a circle of radius b, with a constant speed v, find a_t and a_n at any time.

5. If a point moves so that a_n is constant, must it move in a circle?

6. Find a_n at any time for a point moving thus: $x = t^2$, $y = 5 - 2\,t^2$. Explain the result.

7. An electron shot into a uniform magnetic field moves so that $a_t = 0$ and $a_n = cv$, continually. Show that its path is a circle whose radius is directly proportional to the initial speed V.

8. A weight is whirled around at the end of a rope 3 ft. long, making one revolution per second. What acceleration toward the center must be supplied continually by the pull in the rope?

9. Like Ex. 8 for a 2 ft. rope and 2 rev. per sec.

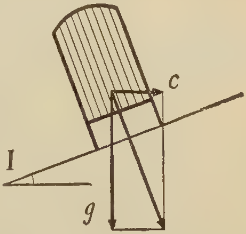
10. When a train running 50 ft./sec. rounds a horizontal curve of radius 1000 ft., what normal acceleration must be supplied by the resistances of the track and roadbed?

11. Like Ex. 10 for 60 mi./hr. and a radius of 1 mi. (Reduce your result to a basis of feet and seconds.)

12. The resultant of the centrifugal and gravitational accelerations c and g, as a train rounds a curve of radius R ft. with a speed v ft./sec. should be perpendicular to the roadbed. Show that the elevation of the outer rail (E in.) should be approximately

$$E = 12\,d\,v^2/gR, \tag{67}$$

where d feet is the distance between the centers of the rails.



13. If $d = 4.92$, V mi./hr. is the speed, and D is the curvature in degrees per hundred feet, show that (67) reduces to $E = .00069\,DV^2$.

§ 286. Motion in Polar Coördinates. In Astronomy, and sometimes elsewhere, the polar coördinates of a moving point P are more convenient than the rectangular. The actual motion is then regarded as composed of one motion along the radius vector and another perpendicular thereto. The "radial motion" tends to change r; the "transverse motion" tends to turn P circularly about O.

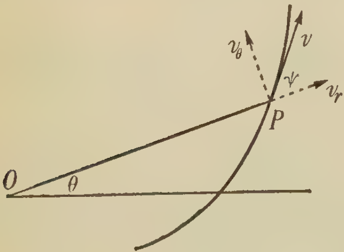


Fig. 174.

Hence we define as component speeds:

$$v_r = \frac{dr}{dt}, \qquad v_\theta = r\frac{d\theta}{dt}. \tag{68}$$

Combining these by the parallelogram method gives for the actual speed and direction:

$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}. \tag{69}$$

$$\tan\psi = \frac{v_\theta}{v_r} \qquad = \frac{r\,d\theta}{dr} = \frac{r}{dr/d\theta}. \tag{70}$$

The latter result is known to be correct by § 230. And (69) follows from § 81 (B), or *Intro.*, p. 400. For

$$ds^2 = dr^2 + r^2\,d\theta^2, \tag{71}$$

and, on dividing by dt^2 and putting $ds/dt=v$, we get $v^2=(dr/dt)^2+r^2(d\theta/dt)^2$, as in (69).

§ 287. Polar Components of Acceleration. Defining the actual acceleration $\vec{a}$ as formerly, we still have $a=\sqrt{a_x{}^2+a_y{}^2}$. But we now need the components of $\vec{a}$ along and perpendicular to the radius vector. These are called the "radial acceleration," a_r, and the "transverse acceleration," a_θ. Each is equal to a multiplied by the cosine of the angle between $\vec{a}$ and the component in question. Multiplying out, as in § 288 below, we find that

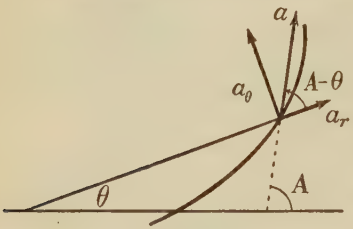


FIG. 175.

$$a_r=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2, \tag{72}$$

$$a_\theta=r\frac{d^2\theta}{dt^2}+2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right). \tag{73}$$

By these formulas we can calculate a_r and a_θ from the equations of motion which give r and θ as functions of t. Then, combining a_r and a_θ by the parallelogram method will give the magnitude of the actual acceleration $\vec{a}$, and the angle which $\vec{a}$ makes with the radius vector.

Remark. In circular motion the speed v equals $rd\theta/dt$, and the normal acceleration a_n necessary to balance the "centrifugal force" is v^2/r or $r(d\theta/dt)^2$, the last term in (72) above. The first term d^2r/dt^2 is what the acceleration would be (outward) if the motion were purely radial, *i.e.*, without rotation. Thus the actual acceleration a_r along the radius vector is the difference of the outward value for radial motion and the inward value for rotation.

§ 288. **Derivation of (72) and (73).** To establish the formulas above we first get the rectangular components a_x and a_y in terms of r and θ. Always

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Differentiating twice with respect to t, and simplifying:

$$\begin{aligned} a_x &= \frac{d^2x}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\cos\theta - \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\sin\theta, \\ a_y &= \frac{d^2y}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\sin\theta + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\cos\theta. \end{aligned} \tag{74}$$

But
$$\begin{aligned} a_r &= a\cos(A-\theta), \\ &= a\cos A\cos\theta + a\sin A\sin\theta. \end{aligned}$$

And, since $a\cos A = a_x$ and $a\sin A = a_y$, this becomes

$$a_r = a_x\cos\theta + a_y\sin\theta. \tag{75}$$

Substituting in (75) the values of a_x and a_y above, and multiplying out, we get the value of a_r in (72).

Similarly to get a_θ multiply a by the cosine of the complement of $(A-\theta)$; that is, by $\sin(A-\theta)$:

$$a_\theta = a_y\cos\theta - a_x\sin\theta. \tag{76}$$

Substituting for a_x and a_y and simplifying:

$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt},$$

as stated in (73).

To get a more compact form for a_θ, which is sometimes more convenient, multiply and divide the right member by r:

$$a_\theta = \frac{1}{r}\left[r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt}\right].$$

The quantity within the bracket is seen to be the derivative of the product $r^2(d\theta/dt)$. Thus we may write

$$a_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right), \tag{77}$$

a form much used in Astronomy. (See §§ 289–290.)

Ex. I. A point moved so that

$$r=t^2, \quad \theta=.1\,t \text{ (radians)}. \tag{78}$$

Find its position, velocity and acceleration at $t=5$; also its path.

Its position was $r=25$, $\theta=.5^{(r)}$. Moreover, by (68):

$$v_r=\frac{dr}{dt}=2\,t, \quad v_\theta=r\frac{d\theta}{dt}=.1\,t^2.$$

At $t=5$, $v_r=10$, $v_\theta=2.5$; whence

$$v=\sqrt{10^2+2.5^2}=\sqrt{106.25}; \quad \tan\psi=\frac{2.5}{10}=.25.$$

The point was moving with a speed of 10.3, in a direction making an angle ψ = 14°, approx., with its radius vector (found above).

$$a_r=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=2-.01\,t^2=1.75,$$

$$a_\theta=r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}=2(2\,t)(.1)=2.$$

The magnitude of the actual acceleration, and the angle B between it and the radius vector, are approximately:

$$a=\sqrt{1.75^2+2^2}=2.66; \quad \tan B=2/1.75, \quad B=49°.$$

To eliminate t from (78) we find $t=10\,\theta$; thus $r=100\,\theta^2$. The path is a spiral, starting out at $\theta=0$.

EXERCISES

1. For each following motion find the radial and transverse components and the total velocity and acceleration, at $t=1$; also the position then, and the polar equation of the path. Draw vectors.

(a) $r=t^2$, $\theta=2\,t^2$;
(b) $r=e^t$, $\theta=3\,t$;
(c) $r=t^2$, $\theta=t^3$;
(d) $r=t^2$, $\theta=4\,t$;
(e) $r=20$, $\theta=4\,t$;
(f) $r=10\cos t$, $\theta=t$;
(g) $r=t+1$, $\theta=1/(t+1)$;
(h) $r=\sqrt{t^2+1}$, $\theta=2\tan^{-1}t$.

2. A point moves in the hyperbola $r = 30/(1+2\cos\theta)$, in such a way that $d\theta/dt = 5/r^2$ constantly. Find the velocity and acceleration at the upper end of the latus rectum. [Hint: Replace $d\theta/dt$ by $5/r^2$ whenever it enters; and simplify each derivative before differentiating further.]

3. Like Ex. 2 for a planet moving in the ellipse $r = p/(1+e\cos\theta)$, with $r^2 d\theta/dt = h$, a constant.

4. In Ex. 3 find the numerical values of the velocity and acceleration in the case of the earth. Here $p = .99974$, $e = .01613$, and $h = .0172$. (The unit of distance is 93 000 000 mi., and the unit of time is 1 day.) Show that v and a are approximately equivalent to 18 mi./sec. and .2 in./sec².

5. If a comet moves in the parabola $r = .5 \sec^2(\theta/2)$, with $r^2 d\theta/dt = .13$, find its velocity and acceleration at the instant when nearest to the sun (*i.e.*, nearest to the pole).

6. If a point moves in any curve so that $r^2 d\theta/dt$ is constant, show that the acceleration is always directed toward the pole; and conversely.

7. Verify the values of a_x and a_y in (74). Also derive (72) in detail from (75) and (73) from (76).

§ 289. Laws of Planetary Motion. Our modern scientific conception of the physical universe began with the astronomical work of Copernicus and Galileo, and was rendered secure by Newton's epoch-making discovery of the law of gravitation. And this was based upon three laws of planetary motion discovered by Kepler.

Kepler's Laws

(I) Relative to the sun each planet moves in an ellipse with the sun at one focus.

(II) The radius vector from the sun to each planet sweeps over equal areas in equal times.

(III) The time of revolution varies as $a^{\frac{3}{2}}$, where a is the major semi-axis of the orbit.

These relations are very nearly exact. There are minor deviations, attributed to the action of the planets upon one another.

Newton's Deductions

(*A*) From Law II it follows that the resultant acceleration, and hence the resultant force, applied to each planet is constantly directed toward the sun.

Proof: The polar element of area $r\,dr\,d\theta$, integrated once, gives $dA = \frac{1}{2} r^2\,d\theta$ as the area swept over by the radius vector in an angle $d\theta$ or time dt. As dA/dt is constant, we have

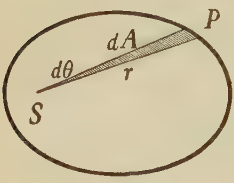


FIG. 176.

$$r^2\,d\theta/dt = h, \text{ some constant.} \tag{79}$$

By Ex. 6, p. 453, this proves Newton's conclusion.

(*B*) From Law I it follows that the acceleration varies inversely as the square of the planet's distance r.

To prove this, simply calculate a_r from (72), as in Ex. 3, p. 453. Starting with the equation of any ellipse,

$$r = \frac{p}{1 + e\cos\theta}, \tag{80}$$

we differentiate with respect to t, replace $d\theta/dt$ by h/r^2, and get

$$\frac{d^2r}{dt^2} = \frac{h^2 e\cos\theta}{pr^2}, \tag{81}$$

$$a = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{h^2}{r^2}\left[\frac{e\cos\theta}{p} - \frac{1}{r}\right] = -\frac{h^2}{pr^2}. \tag{82}$$

Thus, along each given orbit, the acceleration varies inversely as r^2.

(*C*) From Law III it can be shown that the same gravitational constant appears in the attraction of the sun upon each planet. This led Newton to enunciate the law of gravitation for the entire universe.

§ 290. The Inverse Problem. Given, conversely, that the acceleration of a planet or other body is always directed toward a fixed origin and varies inversely as the square of the radius vector, just how will the body move?

We are now given $a_\theta=0$ and $a_r=-k/r^2$; that is,

$$r^2\, d\theta/dt=h, \tag{83}$$

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-\frac{k}{r^2}. \tag{84}$$

Replacing $d\theta/dt$ by h/r^2 in (84) gives

$$\frac{d^2r}{dt^2}=\frac{h^2}{r^3}-\frac{k}{r^2}. \tag{85}$$

This differential equation (85) is of the type treated in § 211. The method there explained gives [cf. Ex. 4, p. 456]:

$$\left(\frac{dr}{dt}\right)^2=-\frac{h^2}{r^2}+\frac{2\,k}{r}+C. \tag{86}$$

To find the relation between r and t, we would separate variables and integrate again. Instead, let us simply determine the path.

Dividing $(dr/dt)^2$ by $(d\theta/dt)^2$ will give $(dr/d\theta)^2$. By (86) and (83) this becomes

$$\left(\frac{dr}{d\theta}\right)^2=\frac{r^2}{h}(Cr^2+2\,kr-h^2). \tag{87}$$

Separating variables and using (42), p. 495:

$$\sin^{-1}\frac{kr-h^2}{r\sqrt{k^2+h^2C}}=\theta+C'. \tag{88}$$

Solving (88) for r gives:

$$r=\frac{h^2/k}{1-(\sqrt{k^2+h^2\,C}/k)\sin(\theta+C')}. \tag{89}$$

This is the equation of a conic whose eccentricity is

$$e=\frac{\sqrt{k^2+h^2C}}{k}=\sqrt{1+\frac{h^2C}{k^2}}, \tag{90}$$

and whose transverse axis is rotated $\pi/2-C'$.

The path is an ellipse, parabola, or hyperbola, according as $e<1$, $e=1$, or $e>1$; that is, according as C is negative, zero, or positive. By (86) this depends upon the speed at any point of the path.

Remark. We do not see the sun attract the earth. We simply infer that it does so, because (aside from minute, explainable discrepancies) the earth moves *just as it would move if attracted by the sun.*

EXERCISES

1. Verify equations (81) and (82), § 289.

2. A sun dial keeps faulty time because the direction of the sun (as seen from the earth against the background of distant stars) changes non-uniformly, — most rapidly when the earth is nearest the sun. Explain this by (79) and Fig. 176.

3. Show that the total solar energy received by the earth from sunshine, per unit angular travel along the orbit, is constant. [Hint: The quantity dQ received in time dt is inversely proportional to the value of r^2 at the instant.]

4. Obtain (86) from (85). Also differentiate (86) as a check.

5. Obtain (88) from (87) by separating variables directly. Also differentiate (88) and check. Also obtain (89) from (88).

6. Find the elastic curve of a cantilever steel beam, 1 in. $\times$ 8 in., carrying a load of 2000 lb. at its free end ($x=100$) in addition to its own weight of 2.2 lb. per in.

7. A point moved thus: $x=t^2$, $y=t^4$. What was its path? Find the position, velocity, acceleration, tangential acceleration and normal acceleration at $t=1$.

8. Show that, for a planet to travel in an ellipse with the sun at the center, and with $r^2 d\theta/dt=h$, the acceleration imparted to it by the sun would have to vary directly as r.

9. Plot $X=3\,t^2$, $Y=2\,t^3$, from $t=-4$ to 4. Find the slope at any point P, and the length of arc from the cusp O to P. Also draw by inspection the involute obtained by unwinding a thread along the upper branch ($t>0$), if the original free length of thread at $t=0$ is 2. Derive equations for that involute, by a method similar to that used for the circle in *Intro.*, § 274. Find each rectangular equation, for the given curve and for the involute. (Cf. Fig. 162, p. 431.)

10. Derive the equations of the evolute of the final curve in Ex. 9. Compare with the equations first given.

CHAPTER XII

CURVES AND SURFACES

PART I. PLANE CURVES. ENVELOPES

To conclude this course let us consider a few more formulas and processes relating to curves and surfaces. We begin with curves in a plane, — say, the plane $z=0$. A single equation $f(x, y)=0$ then represents a curve.

§ 291. **Families of Curves.** For any value of k the equation $x^2+y^2=k^2$ represents a certain circle, with center (0, 0). The total set of all such circles, for all values of k, is called a *family* of circles, and k is called the *parameter* of the family. In general, a family of curves is a class of curves having some common geometric property, or a common type equation, and differing only as to the value assigned to one or more arbitrary constants or parameters in the equation.

This meaning of "parameter" while different from the former one is similar. Each value given the parameter in a pair of parametric equations (§ 15) determines a *point;* changing it moves the point. Each value given the parameter in the equation of a "family" fixes a *curve;* changing it moves (*i.e.*, modifies) the curve.

§ 292. **Envelopes.** Sometimes a family of curves, $f(x, y, k)=0$ has an "envelope"; *i.e.*, another curve which at each of its points is tangent to some curve C of the family, and which somewhere touches every curve C.

Each k gives a curve C and a point of tangency (X, Y), which lies on the envelope E as well as on C. Going along E, both X and Y change; also k. But at every point:

$$f(X, Y, k)=0, \qquad \frac{dY}{dX}=\frac{dy}{dx}. \tag{1}$$

Since X and Y vary with k along E, the total derivative of $f(X, Y, k)$ with respect to k is

$$\frac{\partial f}{\partial X}\frac{dX}{dk}+\frac{\partial f}{\partial Y}\frac{dY}{dk}+\frac{\partial f}{\partial k}=0, \tag{2}$$

provided these derivatives exist and are continuous. Moreover, along each C, we have everywhere $f(x, y, k)=0$, with k remaining fixed.

$$\therefore \qquad \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}. \tag{3}$$

At the point of tangency (X, Y), (3) becomes $dy/dx=-\partial f/\partial X \div \partial f/\partial Y$. And by (1) this must also equal dY/dX.

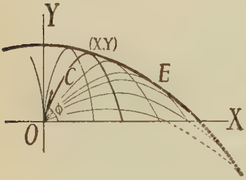


Fig. 177.

$$\therefore \qquad dY=-\frac{\partial f/\partial X}{\partial f/\partial Y}dX. \tag{4}$$

Substituting this value for dY in (2) and reducing, we find that at every point (X, Y) of the envelope:

$$\frac{\partial f}{\partial k}=0. \tag{5}$$

The equations $f(X, Y, k)=0$ and (5) give X and Y in terms of k, as parametric equations of the envelope. Eliminating k will give the ordinary rectangular equation.

Ex. I. Find the envelope, if any, of all possible paths for a ball batted from a point O, in a given vertical plane, with an initial speed of 200 ft./sec. (Ignore air resistance, and take $g=32$.)

Any initial inclination ϕ (Ex. 6, p. 459) gives the path:

$$y=x\tan\phi-.0004\,x^2\sec^2\phi. \tag{6}$$

To find the envelope for all values of ϕ, we differentiate partially with respect to ϕ, as in (5):

$$0=x\sec^2\phi-.0008\,x^2\sec^2\phi\tan\phi. \tag{7}$$

This gives $\tan\phi = 1250/x$. Substituting in (6), using the relation $\sec^2\phi = 1 + \tan^2\phi$, and reducing:

$$y = 625 - .0004\,x^2, \tag{8}$$

i.e.,

$$x^2 = -2500(y - 625). \tag{9}$$

Thus the envelope is a parabola, of focal distance $p = 625$, having its vertex at (0,625) and hence its focus at O.

Remark. Paths C for values of ϕ under 45° would be tangent below the level of O, as shown dotted in Fig. 177. Points outside of E could not be reached with any ϕ for the given initial speed of 200 ft./sec.; points inside can be reached with two different ϕ's. (The dotted portions would be actual paths, if O were at the top of a building or the edge of a cliff.) The envelope E is called the "parabola of surety" for the "trajectories" or paths in question.

EXERCISES

1. What family of curves is represented by each following equation? What is common to all curves of the family? How do the curves change with k?

(*a*) $y = 2\,x + k$, (*b*) $y = kx + 2$,
(*c*) $y^2 = kx$, (*d*) $x^2 + y^2 = kx$,
(*e*) $\dfrac{x^2}{100} + \dfrac{y^2}{k^2} = 1$, (*f*) $\dfrac{x}{10} + \dfrac{y}{k} = 1$.

2. Write the equation for a family of circles with centers on the Y-axis and passing through the origin.

3. Find the equation of a family of circles with centers on the line $y = 2\,x$, and tangent to the X-axis.

4. Find the envelope of each following family of curves. [First try to discover the envelope by experiment, drawing curves of the family roughly for several values of the parameter k, or h, etc.]

(*a*) $x^2 + (y - k)^2 = 25$, (*b*) $(x - h)^2 + y^2 = 16$,
(*c*) $x^2 + (y - k)^2 = \frac{1}{4}\,k^2$, (*d*) $(x - h)^2 + y^2 = 4\,h - 4$,
(*e*) $y^2 = 4\,p(x - p)$, (*f*) $y^2 = 4\,p^3(x - \frac{3}{4}\,p)$,
(*g*) $y = lx + 5/l$, (*h*) $(y - lx)^2 + 100\,l = 0$.

5. Find the envelope of all the ellipses $x^2/h^2 + y^2/k^2 = 1$ which have the area 20π. [Hint: What value must k have in terms of h?]

6. By proceeding as in *Intro.* §§ 191, 223, derive (6).

7. Derive the equation corresponding to (6) for any initial speed v, keeping the gravitational constant as g instead of 32. Also find the envelope corresponding to (8) for this general case.

§ 293. **Related Parameters.** The equation of a family of curves may involve two or more parameters, related in such a way that only one can vary independently. To find the envelope in such a case, we may first eliminate all but one parameter. Or, if this is difficult, we may proceed as in the following example.

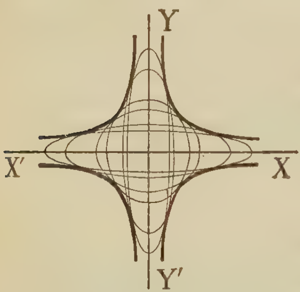


FIG. 178.

Ex. I. Find the envelope of the family of ellipses

$$\frac{x^2}{h^2}+\frac{y^2}{k^2}=1 \qquad (10)$$

having a constant area A. (Fig. 178.)

Here $$\pi hk=A. \qquad (11)$$

Differentiating both (10) and (11) with respect to h:

$$-\frac{2\,x^2}{h^3}-\frac{2\,y^2}{k^3}\frac{dk}{dh}=0, \qquad (12)$$

$$\pi h\frac{dk}{dh}+\pi k=0. \qquad (13)$$

Substituting in (12) the value of dk/dh given by (13):

$$\frac{x^2}{h^2}=\frac{y^2}{k^2}. \qquad (14)$$

Hence by (10): $x^2/h^2=\frac{1}{2}$, or $h=\pm\sqrt{2}\,x$. Also $k=\pm\sqrt{2}\,y$. In (11) these values of h and k give $\pm 2\,\pi xy=A$.

$$\therefore \qquad xy=\pm A/2\,\pi. \qquad (15)$$

The envelope is a pair of rectangular hyperbolas.

§ 294. **Caustic Curves.** Rays of light reflected from most surfaces will not come to a focus anywhere. Often, however,

they will run tangent to some other surface, and will outline this "caustic surface" clearly in any translucent medium. Or, where such a surface meets any material surface a bright curve will appear.

A common example is the pattern often seen on the bottom of a teacup, produced by reflection from the inner surface of the cup. Fig. 179 shows such a "caustic curve" when rays originally almost parallel are reflected from a cylinder.

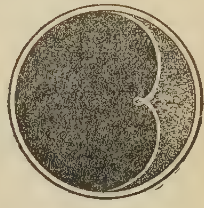


FIG. 179.

When the light source is virtually in the plane in question, the caustic curve is virtually the envelope of rays reflected in that plane. Let us find that envelope when the original rays are horizontal and are reflected from the right semi-circle in Fig. 180.

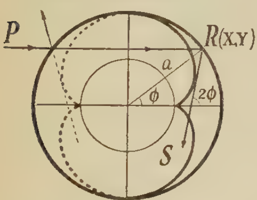


FIG. 180.

The point of reflection is $X=a\cos\phi$, $Y=a\sin\phi$. The inclination of a reflected ray is 2ϕ, and its equation is

$$y-Y=\tan 2\phi(x-X), \tag{16}$$

i.e.,

$$y-a\sin\phi=\frac{\sin 2\phi}{\cos 2\phi}(x-a\cos\phi). \tag{17}$$

Clearing of fractions, transposing terms, and replacing $\sin 2\phi \cos\phi-\cos 2\phi\sin\phi$ by $\sin(2\phi-\phi)$ gives

$$x\sin 2\phi-y\cos 2\phi-a\sin\phi=0. \tag{18}$$

Varying ϕ gives the family of reflected rays.

To find the envelope, we differentiate partially with respect to ϕ:

$$2x\cos 2\phi+2y\sin 2\phi-a\cos\phi=0. \tag{19}$$

To get x in terms of ϕ, we multiply (18) by $2\sin 2\phi$ and (19) by $\cos 2\phi$, and add. This gives (Ex. 4, p. 462):

$$x=\frac{a}{2}(2\sin 2\phi\sin\phi+\cos 2\phi\cos\phi). \tag{20}$$

Or, using the double-angle formulas, this reduces to

$$x=a(\tfrac{3}{2}\cos\phi-\cos^3\phi). \tag{21}$$

Further, by the formula for $\cos 3\phi$, p. 490, we see that $\cos^3\phi=\frac{3}{4}\cos\phi+\frac{1}{4}\cos 3\phi$, so that (21) gives

$$x=\frac{a}{4}(3\cos\phi-\cos 3\phi). \tag{22}$$

Again, multiplying (18) by $-2\cos 2\phi$ and (19) by $\sin 2\phi$, adding, and reducing similarly, we find

$$y=\frac{a}{4}(3\sin\phi-\sin 3\phi). \tag{23}$$

Equations (22) and (23) suggest an epicycloid. And, by comparison with (28), p. 31, they are seen to represent such a curve, — with two cusps. The radii of the rolling circle and fixed circle would be $a/4$ and $a/2$.

Remarks. (1) The dotted portion in Fig. 180 corresponds to rays reflected outward from the left semi-circle. The direction, if reversed, would be tangent to the epicycloid.

(2) The reflected ray RS will later be reflected again, — this time with inclination 4ϕ, at a point

$$X'=a\cos(\pi+3\phi),\ Y'=a\sin(\pi+3\phi). \tag{24}$$

The envelope is again an epicycloid, but different in shape.

EXERCISES

1. The normals to any curve have what relation to the evolute? What new definition could we now give for the evolute of a curve?

2. Find the envelope of the family of lines $x/a+y/b=1$, in which $ab=4$ continually.

3. Find the envelope of a family of straight lines:

(*a*) If the sum of the intercepts of each line on the X- and Y-axes is a constant k;

(*b*) If the triangular area formed with the axes is constant;

(*c*) If the length of each line between the axes is constant.

4. Derive (18) from (17), also (20). Verify (21) and (22).

5. Derive (23) in detail from (18) and (19).

6. Calculate dy/dx from (22) and (23), simplify, and show that the slope of the above epicycloid at any point is $\tan 2\phi$. [Cf. (16).]

PART II. LINES AND CURVES IN SPACE

§ 295. Direction Cosines. The direction of a line L in space is described by giving its "direction angles," *i.e.*, the angles α, β, γ (not exceeding 180°) which it makes with the positive axes OX, OY, OZ, — or with any lines parallel to those axes which it meets. The angles α, β, γ are usually given by their cosines, called the "direction cosines" of L:

$$l = \cos\alpha, \qquad m = \cos\beta, \qquad n = \cos\gamma. \tag{25}$$

A line L is often regarded as running in either of two exactly opposite directions. The angles α, β, γ for one direction are supplementary to the angles for the other; and the direction cosines are therefore numerically equal but opposite in sign.

For a straight line through any two points, — running, let us say, from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$, — the direction cosines have the following simple values:

$$l = \frac{x_2 - x_1}{D}, \qquad m = \frac{y_2 - y_1}{D}, \qquad n = \frac{z_2 - z_1}{D}, \tag{26}$$

where D is the distance between P_1 and P_2, *i.e.*,

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{27}$$

For, by Fig. 181, $(x_2 - x_1)$ is the leg adjacent to α in right triangle P_1LP_2, whose hypotenuse is D. Thus

$$\cos\alpha = (x_2 - x_1)/D, \quad \text{etc.} \tag{28}$$

Theorem. The sum of the squares of the direction cosines of any line is always unity:

$$\boldsymbol{l^2 + m^2 + n^2 = 1.} \tag{29}$$

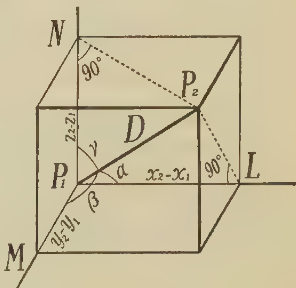


FIG. 181.

For, if we select two points on the line, l, m, n have the values in (26). Squaring and adding gives (29).

In short, then, the three direction cosines of a line through two given points are proportional to the difference of the x's, the difference of the y's, and the difference of the z's: Δx, Δy, Δz.

If we know any three numbers which are *proportional* to the direction cosines of a straight line, the actual cosines will be those three numbers divided by the square root of the sum of their squares. To prove this, let the direction cosines be proportional to a, b, c. Then

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}. \tag{30}$$

Calling the common value of these fractions k, we have

$$l=ka, \qquad m=kb, \qquad n=kc. \tag{31}$$

By (29), $k^2a^2+k^2b^2+k^2c^2=1$, or $k=\pm 1/\sqrt{a^2+b^2+c^2}$.

$$\therefore \qquad l=\pm a/\sqrt{a^2+b^2+c^2}, \quad \text{etc.} \tag{32}$$

§ **296. Equations of a Straight Line.** The position of a line in space is determined by any point (x_1, y_1, z_1) through which it passes, and its direction angles, α, β, γ.

If (x, y, z) be any point on the line, then, by (26):

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=D. \tag{33}$$

These equations hold for every point on the line, and for no others. Hence they are the equations of the line.

If the line be regarded as running *from* (x_1, y_1, z_1) *to* (x, y, z), then D is to be measured positively in that direction.

§ **297. Angle between Two Directions.** The angle A between any two lines drawn from a point $Q(X, Y, Z)$ in space is given by a simple formula:

$$\cos A=ll'+mm'+nn', \tag{34}$$

where l, m, n and l', m', n' are the direction cosines of the two lines.

To prove this, consider points $P(x, y, z)$ and $P'(x', y', z')$ on the lines, a unit distant from Q. By (33), $x = X + l$, etc. Thus

$$x = X + l, \qquad y = Y + m, \qquad z = Z + n, \tag{35}$$

$$x' = X + l', \qquad y' = Y + m', \qquad z' = Z + n'. \tag{36}$$

By (25), p. 205, the square of the distance between P and P' is

$$d^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 = (l - l')^2 + (m - m')^2 + (n - n')^2.$$

Squaring out, and remembering that the sum of the squares of the direction cosines for each line is unity, we get

$$d^2 = 2 - 2(ll' + mm' + nn'). \tag{37}$$

But, by the Cosine Law, we have directly from Fig. 182:

$$d^2 = 1^2 + 1^2 - 2(1)(1)\cos A. \tag{38}$$

Comparing (37) and (38), we have (34) above.

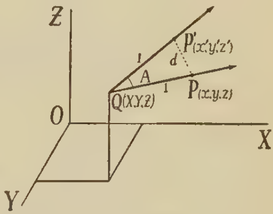


FIG. 182.

Thus, we can find the cosine of the angle between two intersecting lines by merely multiplying together the corresponding direction cosines in pairs, and adding.

If two lines in space do not lie in a common plane and hence do not meet, we still speak of the angle "between" them, — meaning the angle between their *directions*, or between parallels to the given lines through some point.

EXERCISES

Here l, m, n denote the direction cosines of a straight line.

1. Find n, and also the direction angles α, β, γ, if

(*a*) $l = \frac{1}{2}$, $m = \frac{1}{2}$; (*b*) $l = .6$, $m = .8$; (*c*) $l = 0$, $m = -\frac{1}{2}$.

2. Find γ if $\alpha = 45°$ and $\beta = 60°$.

3. Find α, β, γ if l, m, and n are proportional to the values:

(*a*) 9, 12, and 20; (*b*) 1, 1, and -1; (*c*) 1, -2, and 3.

4. Find l, m, n for the line joining (5, 2, 0) and (8, 1, 6).

5. Find the equations of the straight line through (2, 0, 4), whose direction cosines are proportional to -3, 4, and 12. Then find two more points on the line.

6. Like Ex. 5 for the line joining (2, 0, 4) and (6, 12, 1). Find the points where the line meets the planes $x=4$ and $z=0$.

7. Find the angle between two lines whose direction cosines are proportional to 2, 5, 1, and 3, 0, -2, respectively.

8. Find the angle between the lines joining (0, 0, 5) to (1, 2, 6) and to (12, -4, 8). Also, those joining (1, 2, 3) to (3, 1, 8) and (4, 13, 4).

9. Find l, m, n for the line whose equations are $(x-2)/3=(y-7)/2=(z+4)/5$. Solve by inspection; also by finding two points.

10. Find l, m, n for the intersection of each pair of planes:

(*a*) $x-2z=3$, $y+z=8$; (*b*) $x+y+z=4$, $x-y+2z=5$.

11. For a line in the XY-plane verify (29) by trigonometry.

12. What is the algebraic condition for two lines in space to be perpendicular? Show how, in the XY-plane, this reduces to the slope test.

§ 298. Twisted Curves: Length and Direction.

A curve which does not lie in any one plane is called a "twisted" or "skew" curve. Along it both y and z vary with x in some definite way, or all three vary with any suitable parameter ϕ:

$$x=f_1(\phi), \qquad y=f_2(\phi), \qquad z=f_3(\phi). \tag{39}$$

If the curve is given as the intersection of two surfaces, it is represented by their simultaneous equations:

$$F_1(x, y, z)=0, \qquad F_2(x, y, z)=0. \tag{40}$$

This form (40) can be changed to (39) by putting x equal to some convenient function of an arbitrary parameter ϕ, and solving (40) for y and z in terms of ϕ.

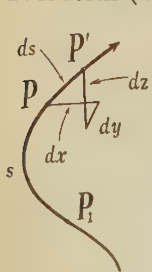


Fig. 183.

Of course, the intersection of two surfaces may be a plane curve in space. But the same procedure will apply, unless the curve lies in some plane $x=c$. In the latter case, put y equal to some function of ϕ and solve for z.

Let s be the length of a twisted curve from a fixed point P_1 to a variable point P; and ds the length of an infinitesimal arc from P to $P'(x+dx,$

$y+dy,\ z+dz$). Regarding arc PP' as straight, we see by (27) that

$$ds = \sqrt{dx^2+dy^2+dz^2}. \tag{41}$$

If parametric equations such as (39) are given, dx, dy, dz are known in terms of ϕ and $d\phi$, and we have

$$s = \int \sqrt{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 + \left(\frac{dz}{d\phi}\right)^2}\, d\phi. \tag{42}$$

In particular, ϕ might be one of the three coördinates, x, y, z.

Further, by § 295, the direction cosines of the arc ds or its tangent line are given by

$$l = \frac{dx}{ds}, \qquad m = \frac{dy}{ds}, \qquad n = \frac{dz}{ds}. \tag{43}$$

We get dx, dy, dz in terms of $d\phi$, and find ds by (41).

More accurately stated: The chord PP' (Fig. 184) has the direction cosines

$$l' = \frac{\Delta x}{c}, \quad m' = \frac{\Delta y}{c}, \quad n' = \frac{\Delta z}{c}. \tag{44}$$

The direction cosines of the tangent are the respective limits of these fractions as the arc $\Delta s \to 0$. But we may write

$$l' = \frac{\Delta x}{\Delta s} \cdot \frac{\Delta s}{c}, \quad \text{etc.} \tag{45}$$

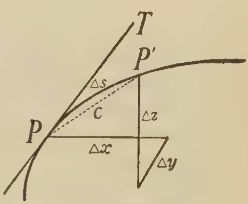


Fig. 184.

And the limit of $\Delta x/\Delta s$ is dx/ds, while the ratio of arc to chord $(\Delta s/c)$ approaches unity for any ordinary curve. Thus we get (43).

Again, strictly speaking, the length of the arc s is the *limit* as $\Delta\phi \to 0$ of the sum of the lengths of its chords:

$$\left[\sqrt{\left(\frac{\Delta x}{\Delta\phi}\right)_1^2 + \left(\frac{\Delta y}{\Delta\phi}\right)_1^2 + \left(\frac{\Delta z}{\Delta\phi}\right)_1^2}\,\Delta\phi + \sqrt{\left(\frac{\Delta x}{\Delta\phi}\right)_2^2 + \left(\frac{\Delta y}{\Delta\phi}\right)_2^2 + \left(\frac{\Delta z}{\Delta\phi}\right)_2^2}\,\Delta\phi + \cdots\right].$$

And, by § 58, this limit can be shown to give (42).

Remark. Strictly, the word "infinitesimal" should be used only in a limit sense: a *variable* approaching zero. An "infinitesimal arc" is considered straight since its maximum deviation from a straight line in proportion to its length approaches zero with Δs.

§ 299. Motion in Space. As a point moves in space in any way, its coördinates vary with the time t:

$$x=F_1(t), \qquad y=F_2(t), \qquad z=F_3(t). \tag{46}$$

We may regard (46) as the "equations of motion," or as parametric equations of the path. [Cf. (39).]

The axial components of velocity are $v_x=dx/dt$, etc.; and the corresponding components of acceleration are $a_x=d^2x/dt^2$, etc. The total velocity and acceleration are directed quantities in space, with the magnitudes

$$v=\sqrt{v_x^2+v_y^2+v_z^2}, \quad a=\sqrt{a_x^2+a_y^2+a_z^2}, \tag{47}$$

and with direction cosines equal respectively to

$$\frac{v_x}{v}, \frac{v_y}{v}, \frac{v_z}{v}, \text{ and } \frac{a_x}{a}, \frac{a_y}{a}, \frac{a_z}{a}. \tag{48}$$

EXERCISES

1. Find the length of the curve $x=3\cos\theta$, $y=3\sin\theta$, $z=4\theta$, from $\theta=0$ to $\theta=\pi/2$. Also find the direction at $\theta=\pi/4$.

2. In Ex. 1 write the equations of the tangent line at $\theta=\pi/4$. At what point does this tangent meet the plane $z=0$?

3. (*a*), (*b*). Like Ex. 1–2 for the curve $x=10\cos t$, $y=10\sin t$, $z=5t$, at $t=\pi/3$.

4. Show that the curve in Ex. 1 lies on a certain cylinder.

5. Find the length of $x=2\theta\cos\theta$, $y=2\theta\sin\theta$, $z=4\theta$, from $\theta=0$ to $\theta=2$. Also show that the curve lies on a certain cone.

6. In Ex. 5 find the equations of the tangent line at $\theta=\pi/2$.

7. A point moves so that $x=t^2$, $y=t^3$, $z=t+2$. Find the direction and magnitude of the velocity and acceleration at $t=1$.

8. Like Ex. 7 for $x=8\sin t$, $y=6\sin t$, $z=10\cos t$, at $t=\pi/4$. Also find the distance traveled from $t=0$ to $\pi/4$.

9. Find the direction and magnitude of the acceleration at $t=\pi/2$, if a point moved as in Ex. 3, with t denoting time.

10. Find parametric equations for the intersection of $z=x^2+y^2$ and $y=2x$. Find the direction of the tangent at (1, 2, 5).

11. Like Ex. 10 for $z=x^2+y^2$, $x=2$, and the point (2, 3, 13).

12. Find the angle between two lines whose equations are, respectively:

$$x=y-3=(z+1)/4; \text{ and } y+z=8,\ x=3.$$

§ 300. **Equation of a Plane.** Let $N(X, Y, Z)$ be the foot of the normal from the origin to any plane (Fig. 185). Let l, m, n be the direction cosines of the normal, and p be its length. Then, by (33),

$$X = pl, \qquad Y = pm, \qquad Z = pn. \tag{49}$$

If $P(x, y, z)$ be any point in the plane, the line NP is perpendicular to ON. Also, the direction cosines of NP are proportional to $(x-X)$, $(y-Y)$, and $(z-Z)$. Hence, by (34),

$$l(x-X)+m(y-Y)+n(z-Z)=0. \tag{50}$$

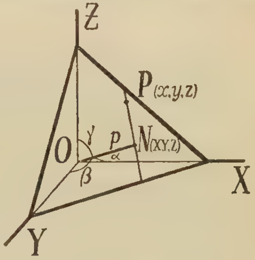


FIG. 185.

Multiplying out, substituting for X, Y, Z from (49), reducing, and transposing, we get

$$\boldsymbol{lx+my+nz=p}. \tag{51}$$

This is the equation of any plane. Observe that l, m, n, and p refer not to the plane itself but to its normal. [That any linear equation represents some plane can be seen by comparing the equation with (51): a plane can be chosen which will have the given equation.]

If the length of the normal is unknown, but some point (x_1, y_1, z_1) in the plane is known, we can find from (51) that $p = lx_1 + my_1 + nz_1$; and (51) then reduces to

$$l(x-x_1)+m(y-y_1)+n(z-z_1)=0. \tag{52}$$

Ex. I. Find the equation of the plane through $(7, 4, -2)$, whose normal makes equal angles with the positive axes.

Here $l=m=n$; and, since $l^2+m^2+n^2=1$, we have $3\,l^2=1$. Thus $l=m=n=\pm 1/\sqrt{3}$. Hence, by (52),

$$(x-7)+(y-4)+(z+2)=0,$$

or

$$x+y+z=9. \tag{53}$$

Ex. II. Find the distance of the plane $2\,x-3\,y+5\,z=12$ from the origin; also the direction cosines of the normal.

Here l, m, n of (51) must be proportional to the coefficients 2, -3, 5; and equal to these divided by $\sqrt{2^2+(-3)^2+5^2}$, or $\sqrt{38}$. Dividing both sides of the given equation by $\sqrt{38}$, it takes the form (51), and we see that $p=12/\sqrt{38}$.

§ 301. Special Planes for Curves and Surfaces.

(I) The plane normal to a given curve C at any point $P_1(x_1, y_1, z_1)$ is perpendicular to the tangent line at P_1. Its equation can be written by (52), taking l, m, n proportional to dx, dy, dz along C.

(II) The *plane tangent to a given surface*

$$F(x, y, z)=0, \tag{54}$$

at any point $P_1(x_1, y_1, z_1)$, is

$$\frac{\partial F}{\partial x_1}(x-x_1)+\frac{\partial F}{\partial y_1}(y-y_1)+\frac{\partial F}{\partial z_1}(z-z_1)=0. \tag{55}$$

Proof. The required equation must be of the form (52). Hence, if $n\neq 0$, we may write it also in the form

$$z-z_1=A(x-x_1)+B(y-y_1). \tag{56}$$

Its intersection with the plane $y=y_1$ is a line, $z-z_1=A(x-x_1)$, which must be tangent at P_1 to the section of the surface made by that plane. The slope of the line is A; of the section, $\partial z_1/\partial x_1$. The latter is found from the equation of the surface (54); and we have

$$A=\frac{\partial z_1}{\partial x_1}=-\frac{\partial F/\partial x_1}{\partial F/\partial z_1}. \tag{57}$$

Likewise

$$B=\frac{\partial z_1}{\partial y_1}=-\frac{\partial F/\partial y_1}{\partial F/\partial z_1}. \tag{58}$$

Substituting these values in (56) and reducing gives (55).

Remarks. (I) The direction cosines of the *line normal to the surface* are proportional to

$$\partial F/\partial x_1, \qquad \partial F/\partial y_1, \qquad \partial F/\partial z_1. \tag{59}$$

(II) When $n=0$ at P, the discussion above must be modified. By taking the plane $x=0$ as the base plane instead of $z=0$, the same result is obtained. It simply happens that $\partial F/\partial z_1=0$ there.

EXERCISES

1. Write the equation of a plane if its normal from (0, 0, 0) has

(a) $l=\dfrac{1}{\sqrt{6}},\ m=\dfrac{2}{\sqrt{6}},\ n=-\dfrac{1}{\sqrt{6}},\ p=5;$

(b) $\alpha=60°,\ \beta=135°,\ \gamma=90°,\ p=0.$

2. Find the equation of a plane 10 units from the origin, if the direction cosines of its normal are proportional to 3, -4, and 12.

3. Find the equation of a plane through (5, 2, -1), if the normal at this point passes also through (2, 1, 6).

4. Like Ex. 3 for the points (4, 8, 0) and (9, 3, 1), respectively.

5. Find the equation of the plane normal to the curve $x=t^2$, $y=t^3$, $z=t^4$, at the point where $t=1$.

6. Like Ex. 5 for $x=4\cos\theta$, $y=4\sin\theta$, $z=2\theta$, at $\theta=\pi$.

7. Find the equation of the plane tangent to each following surface at the point specified:

(a) $x^2+y^2+2z=14$, at (3, 1, 2);

(b) $z=x^2+4y^2$, at (2, 1, 8);

(c) $x^2+y^2+z^2=24$, at (2, -2, 4);

(d) $z=x^3+y^3$, at (1, 1, 2);

(e) $4x^2+9y^2=25$, at (-2, 1, 20);

(f) $z=xy$, at (-5, -10, 50).

8. (a)–(f). In Ex. 7(a)–(f) find angles α, β, γ for each normal.

9. Find the distance from the origin to each following plane:

(a) $3x-4y+12z=52$; (b) $x+3y-5z=70$.

10. The sphere $x^2+y^2+z^2=100$ is cut by the plane $z=8-.3x-.2y$. Find the radius and area of the common section. [Hint: How far is the plane from the center? Cf. p. 228: (56) and Remark (I).]

§ 303. Parametric Equations of a Surface. Three equations giving the coördinates of a point P as functions of any two independent parameters, ϕ and θ, viz.

$$x=f_1(\phi,\theta),\qquad y=f_2(\phi,\theta),\qquad z=f_3(\phi,\theta),\qquad (60)$$

ordinarily represent a *surface*. For, if we solve two of the equations for ϕ and θ in terms of x and y, and substitute in the third, we shall have a single equation :

$$z = F(x, y). \tag{61}$$

And this, by § 125, generally represents a surface.

On the earth's surface, the position of a point P is described by giving its latitude and longitude, — which we may regard as parameters determining the coördinates (x, y, z) of P.

§ 304. Curves on a Given Surface. In (60), if θ be held constant while ϕ varies, or vice versa, P will trace some special curve on the given surface. In general, however, as P moves on the surface, both ϕ and θ will be changing. Along any one such curve, θ will vary with ϕ in some definite way:

$$f(\phi, \theta) = 0. \tag{62}$$

Thus (60) in effect give x, y, z in terms of ϕ alone, — or θ alone. If the elimination of one parameter is inconvenient, we keep both, together with their relation (62).

Anywhere on a cylinder of radius a and axis OZ, we may take

$$x = a \cos \theta, \qquad y = a \sin \theta, \tag{63}$$

with z an arbitrary function of any other parameter. These equations take the place of (60) for more general surfaces. Elimination of θ gives $x^2 + y^2 = a^2$ as the equation of the cylinder.

Along any curve of the cylinder (except an element or horizontal section), z varies with θ. The relation $f(z, \theta) = 0$ replaces (62).

When studying a curve cut from one surface by another, we may proceed as with (40), § 298.

Ex. I. Write a set of parametric equations for a sphere of radius a and center (0, 0, 0).

Choosing as parameters two spherical coördinates ϕ and θ (Fig. 91, p. 236), we have by (70), § 140,

$$x = a \sin \phi \cos \theta, \quad y = a \sin \phi \sin \theta, \quad z = a \cos \phi. \tag{64}$$

Here θ may be called the longitude, and ϕ the complement of the latitude, for any point on the sphere. (Cf. Fig. 92 *a*, p. 237.)

Ex. II. A curve on the sphere (64) runs in this way:

$$\theta = \log \tan (\phi/2) + \pi/2. \tag{65}$$

Find (*A*) its direction at the "equator" ($\phi = \pi/2$); also (*B*) the angle at which it cuts the meridian $\theta = \pi/3$.

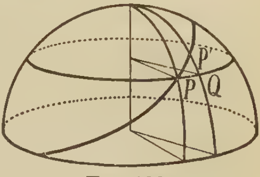


FIG. 186.

(*A*) Differentiating (64) and substituting $\phi = \pi/2$ (and hence also $\theta = \pi/2$), we find

$$dx = -a d\theta, \quad dy = 0, \quad dz = -a d\phi. \tag{66}$$

Also, differentiating (65) and simplifying, we get $d\theta = d\phi/\sin\phi = d\phi$. Thus $dx = dz = -a d\phi$, whence $ds = \pm a\sqrt{2}\, d\phi$. The direction cosines are dx/ds, etc. Hence

$$l = \mp 1/\sqrt{2}, \qquad m = 0, \qquad n = \mp 1/\sqrt{2}. \tag{67}$$

The tangent is perpendicular to the Y-axis, and makes angles of 45° (or 135°) with the positive X- and Z-axes.

(*B*) We could find l, m, n for the above curve at $\theta = \pi/3$, and also for the meridian along which $\theta = \pi/3$ (constantly); and then apply (34), § 297. Instead, let us note the infinitesimal triangle PQP', in which the sides or arcs PQ and QP' are, by inspection: $PQ = a \sin\phi\, d\theta$, $QP' = a\, d\phi$. Or, since $d\theta = d\phi/\sin\phi$, $PQ = a\, d\phi$. Thus the required angle is given by $\tan \angle PP'Q = \overline{PQ}/\overline{QP'} = 1$. The curve crosses this meridian, and, in fact, every meridian at an angle of 45°.

A curve which crosses all meridians at the same angle is called a "loxodrome," or, in navigation, "a rhumb line."

EXERCISES

1. Find the rectangular equation of each following surface, and state what kind of surface it is:

(*a*) $x = \theta + \phi$, $y = \theta - 2\phi$, $z = \phi$;

(*b*) $x = r \cos\theta$, $y = r \sin\theta$, $z = r$;

(*c*) $x = u$, $y = v$, $z = u^2 + v^2$;

(*d*) $x = t$, $y = u$, $z^2 = 1 - t^2 - u^2$;

(*e*) $x = 5 \sin\phi \cos\theta$, $y = 4 \sin\phi \sin\theta$, $z = 3 \cos\phi$.

2. Like Ex. 1 for the following. [Rotate the X- and Y-axes 45°.]

(*a*) $x=r\cos\theta,\ y=r\sec\theta,\ z=r^2$;

(*b*) $x=t,\ y=u,\ z=10\,tu$.

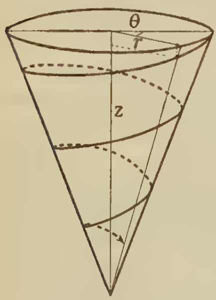
3. Where are all points in space for which $x=a\cos\theta$, $y=a\sin\theta$, $z=\phi$? All such points for which $\phi=10$? All for which $\theta=\pi/4$?

4. Obtain some algebraic parametric equations for the surface $z=x^2-y^2$. Also some with $x=\phi\sec\theta$. Some with $x=t\cosh u$.

5. Solve Ex. II, § 304, for another loxodrome: $\theta=2\log\tan(\phi/2)$.

6. On the cone in Ex. 1(*b*) a curve runs in this way: $r=10-\theta$. Find its length from $\theta=0$ to $\theta=1$; also its direction at $\theta=1$.

7. A helix is a curve which winds around a surface of revolution, while advancing axially at a constant rate per unit angle turned.



Show that a cylindrical helix has the equations $x=a\cos\theta$, $y=a\sin\theta$, $z=k\theta$. Find its length from $\theta=0$ to $\theta=\pi$; also its direction at $\theta=\pi$.

8. Find the equations for a conical helix, on a cone whose vertex angle is 90°, if $dz/d\theta=k$. [Cf. Ex. 1(*b*), 6.]

9. Show that parametric equations of a sphere in terms of the latitude L and longitude θ are: $x=a\cos L\cos\theta$, $y=a\cos L\sin\theta$, $z=a\sin L$.

10. In Ex. 9 find the direction, at $L=\pi/6$, of a great circle along which $z=x$.

§ 305. Ruled Surfaces. It is possible to draw straight lines upon certain of the conicoids besides the cylinder and cone.

Consider, for example, the hyperboloid of one sheet:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1. \tag{68}$$

If we transpose either positive fraction, both sides of the resulting equation can be factored. *E.g.*,

$$\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right). \tag{69}$$

Consider any point in space for which

$$\frac{x}{a}+\frac{z}{c}=k\left(1+\frac{y}{b}\right) \quad\text{and}\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{k}\left(1-\frac{y}{b}\right), \tag{70}$$

where k is any constant. All such points lie on a certain straight line, whose equations are (70). But, when (70) both hold, (68) above is also satisfied. Hence, every such point lies on the given hyperboloid. Therefore, the line represented by the equations (70) is a straight line lying entirely on the surface. This is true for *any* value of k that we may choose.

Through any given point (x_1, y_1, z_1) of the hyperboloid, such a line passes. For, substituting x_1, y_1, z_1 in (70) will determine a value for k or $1/k$. The two are necessarily consistent.

Moreover, if in (70) we pair the factors of (69) differently, we find likewise that the hyperboloid contains the family of lines whose equations are

$$\frac{x}{a}+\frac{z}{c}=m\left(1-\frac{y}{b}\right), \quad \frac{x}{a}-\frac{z}{c}=\frac{1}{m}\left(1+\frac{y}{b}\right). \qquad (71)$$

Through each point (x_1, y_1, z_1) passes one line of each family. [Fig. 75, p. 208, shows a special case of this.]

§ 306. **Conics as Sections of a Cone.** When an ordinary circular cone is cut by a plane, the section is a "conic," as defined in § 239.

Proof. Choose the Y-axis parallel to the cutting plane PQR (Fig. 187). By § 129 the equation of the cone is

$$x^2+y^2=k^2z^2. \qquad (72)$$

If we rotate the X- and Z-axes through an angle A, the inclination of plane PQR, the equation of the latter will be simply $z=c$, some constant. And in (72) x and z will be replaced by $(x \cos A - z \sin A)$ and $(x \sin A + z \cos A)$, respectively. (§ 221.) Thus (72) will become:

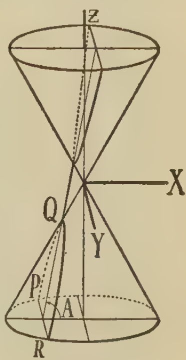


Fig. 187.

$$(x \cos A - z \sin A)^2 + y^2 = k^2(x \sin A + z \cos A)^2. \qquad (73)$$

Hence, at the intersection of the cone with the plane $z=c$:

$$(x \cos A - c \sin A)^2 + y^2 = k^2(x \sin A + c \cos A)^2. \qquad (74)$$

As (74) is of the second degree, its locus (in the specified plane) is some conic. (§ 250.)

Indeed, this is the reason for the name "conic" as applied to the curves of Chapter X. And the various "conicoids" are so called from the fact that all their plane sections are conics.

EXERCISES

1. For the hyperboloid $x^2/25+y^2/16-z^2/9=1$, determine k in (70) so that the line shall pass through (10, 4, 6). Also find the direction cosines of the line.

2. Like Ex. 1 for m in (71), for the second line through (10, 4, 6).

3. Find the equations for two different lines on the hyperboloid $x^2/9 -y^2/4 +z^2 = 1$ through the point (3, 6, -3).

4. Find the equations of two lines on the paraboloid $z=x^2-4\,y^2$ through (5, 2, 9). [Hint: Factor z as $k(z/k)$.]

5. (a), (b). Find the angle between the two lines in Ex. 3; also in Ex. 4.

6. For a cone with a vertex angle of $\pi/2$, what sort of conic will be obtained if $A=\pi/6$ in Fig. 187? If $A=\pi/4$? If $A=\pi/3$? [Try to tell by inspection; but check by (74), using the quick test in §252.]

7. If the cone in Ex. 6 be cut by a plane whose inclination is $\sin^{-1} .6$, and whose distance from the vertex is 10, find the semi-axes of the ellipse cut out.

8. What angle does the line $x=y=z$ make with the normal to the surface $10\,z=x^2+4\,y^2$ at the higher intersection?

9. Like Ex. 8 for $x=2\,y=z$, and $z=xy$.

10. Find the angle between the cylindrical helix $x=a\cos\theta$, $y=a\sin\theta$, $z=k\theta$, and any element, $\theta=\theta_1$, at their intersection.

11. Find the envelope of the lines $y=lx-l^2$. Plot several; check.

12. Find the area of the part of the plane $3\,x-4\,y+12\,z=39$ which lies within the sphere $x^2+y^2+z^2=25$.

13. Show that the general equation of a loxodrome crossing all meridians at an angle A is $\theta \operatorname{ctn} A=\log\tan\phi/2+c$. Compare this with (65).

14. Show that the equation of a plane tangent to a conicoid $ax^2+by^2+cz^2=1$ at (x_1, y_1, z_1) can be written by a rule analogous to that in § 258.

RETROSPECT

In the *Introduction* we defined and studied certain elementary functions (power, trigonometric, exponential and logarithmic), some simple types of motion, and a few curves, mostly conics. We also touched upon some topics of algebra, such as progressions, probability, complex numbers, and the solution of higher equations. But the central purpose of the introductory course, as of the present one, was the solution of the two basic problems of mathematical analysis, — viz. the rate problem and its inverse, — by differentiation and integration.

We have now defined and differentiated further functions (inverse trigonometric, hyperbolic, etc.), and have seen how to integrate all forms that lead to elementary functions, and how to approximate other integrals. By applying differentiation further we have obtained more formulas for the study of motion, a method of evaluating indeterminate forms, further methods of studying a curve (to find its curvature, evolute, singular points, asymptotes, and any special tangent, normal or envelope properties; also to recognize the equation in many cases, or reduce it by rotation, etc.); also some idea of the equations of surfaces and the uses of partial and total derivatives. Likewise we have used integration more extensively, — to calculate mean values, many physical quantities, and some from economics, — and have solved certain elementary types of differential equations.

You will find it profitable to make a careful review, so as to see the course in perspective as a whole, and tie the most important ideas securely together in permanent form. In the light of your present knowledge, select throughout the

course the formulas which seem most important, — the key formulas, — and make a list of these in your permanent notes. Also record useful drawings, such as the differential triangle (from which many formulas can be read off without the use of memory), or some typical sections used in determining limits of integration in working with solids, etc. Most of all, be sure that you are clear as to the *meaning* of important terms (such as curvature, evolute, envelope, acceleration, etc.), and that you can instantly call to mind the general plan of procedure for solving each important kind of problem.

E.g., if we encounter an indeterminate form, no matter in what connection, we ought immediately to recognize it as such, and know that to evaluate it we must reduce it to a fraction ($0/0$ or ∞/∞) and then differentiate numerator and denominator separately. Or, again, to find an envelope of a family $f(x, y, k)=0$, we should know that the first step is to differentiate partially with respect to k and put $\partial f/\partial k=0$. Or, again, in plotting a curve, we should think of cutting it by lines $y=lx$, as well as $y=c$, or $x=c$, to get points. And so on.

We should know that many integrals must be found by approximation; and not waste time trying to integrate exactly forms like

$$\sin x^2\,dx, \quad \sqrt{\sin x}\,dx, \quad \sqrt{1+x^3}\,dx, \quad e^x dx/x, \text{ etc.}$$

And likewise we should never waste time trying to solve exactly an equation involving both algebraic and transcendental functions of an unknown quantity, such as

$$x+\sin x=2, \qquad 3\,x=\log x, \qquad e^x=3\,x, \text{ etc.}$$

The special points just mentioned are simply a few illustrations to make clear the meaning of the general suggestion above.

Most of the topics of the present course are covered far more extensively in more advanced, specialized courses. Particularly prominent in higher analysis is the rôle played by complex variables, partial derivatives, series, and vector methods. These, and certain ideas of modern geometry, have contributed largely to the theory of relativity and other fundamental developments in the field of physics.

EXERCISES FOR GENERAL REVIEW

1. Draw by inspection; or plot by points and by suitable tests:

(*a*) $r=4\cos 5\theta$; (*b*) $r=2\sec(\theta-\pi/3)$;

(*c*) $r=6/(1-\frac{1}{2}\cos\theta)$; (*d*) $5x^2+2xy+y^2=16$;

(*e*) $x=5\cos\phi-\cos 5\phi,\quad y=5\sin\phi-\sin 5\phi$;

(*f*) $x=2\cos\phi+\cos 2\phi,\quad y=2\sin\phi-\sin 2\phi$.

2. Find the polar equation of a circle of radius 5 in. with center $(11, 20°)$. Does the circle meet the line $\theta=50°$?

3. What are the parametric equations of a cycloid generated by a rolling circle of radius 7? Find the rectangular equation.

4. If $mx^2+ny^2=1$ is a homogeneous equation of an ellipse, what "dimensions" must m and n have, in terms of length?

5. Write the differential of each following function:

(*a*) $\log\dfrac{x-2}{x-1}+\dfrac{1}{x-1}$, (*b*) $e^{\sqrt{\frac{1-\operatorname{ctn} x}{1+\operatorname{ctn} x}}}\left(\sqrt{\dfrac{1-\operatorname{ctn} x}{1+\operatorname{ctn} x}}-1\right)$,

(*c*) $k\log(\sqrt{x}+\sqrt{x-k})-\sqrt{x^2-kx}$,

(*d*) $\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}-\dfrac{1}{3}\sec^{-1}\dfrac{\sqrt{x^2+9}}{3}$.

6. (*a*)–(*d*). Check (37), (43), (67), (75), pp. 494–97, by differentiation.

7. In $F=MM'/r^2$, if M, M', and r are taken as 20, 35 and 50, instead of 20.02, 34.97, and 50.04 (their true values), approximately what error will there be in the calculated F?

8. Test the surface $z=x^3-12x-y^2$ for maximum and minimum heights. Draw profiles through any critical points; also contours at corresponding levels.

9. Find the slope of the steepest section of the surface in Ex. 8 at $(4, 6, -20)$; also what direction angle τ gives it.

10. Like Ex. 8 for $z=12r-r^3$ (in cylindrical coördinates).

11. Find the tenth derivative of $4\sin\theta\cos\theta$. [Use any available short cut.]

12. The combined surface area of a sphere and cube is to be minimized while keeping the combined volume constant. Find the ratio of the radius to the edge.

13. In finding the minimum deviation of certain rays of light, it was necessary to get the minimum value of $\phi-\theta$, subject to the condition $\sin\phi-\sin\theta=n\lambda/e$, a constant. Do this.

14. Find the minimum value of $.8\cos\theta+\theta^2-.6\theta$.

15. The best time t for removing a metal tire from a certain heated wheel is the value of t which maximizes $e^{-\frac{x^2}{4k^2t}}/\sqrt{t}$. Find that t.

16. A certain commodity costs \$2 per unit to produce. At any price (\$$x$), there can be sold y units, where $y=800+200\sqrt{4.95-x}$. What x will maximize the profit?

17. Find the area of a parabolic ceiling 200 ft. long, whose sides are 20 ft. lower than the middle and are 80 ft. apart.

18. Find the volume of an ellipsoid of semi-axes 9, 6, and 3.

19. Find the work required to compress superheated steam from $v=.4$ to $v=.3$, if $v=1887\,p^{-1}-1.486\,p^{-.477}$. (Cf. §§ 76, 108.)

20. A vertical plane surface, of area A sq. ft., moving horizontally v ft./sec. in still air, encounters a pressure kAv^2 lb./sq. ft. If a vertical rectangle 20 ft. long and 5 ft. high swings around a vertical axis OQ, 400 ft. from its center, with an angular speed of .3 rad./sec., express: (a) The total force encountered; (b) The torque about OQ.

21. Could the following forms be integrals of the same integrand:

$$\log\frac{\sqrt{t^2+t+1}+t-1}{\sqrt{t^2+t+1}+t+1}+\sin^{-1}(\cos t); \qquad \log\frac{\sqrt{t^2+t+1}-t-1}{\sqrt{t^2+t+1}-t+1}-t?$$

22. Integrate each following quantity:

(a) $\sec^6 x\tan^5 x\,dx$, (b) $\sec^5\phi\,d\phi$,

(c) $\cos^2 t\cos^2 6\,t\,dt$, (d) $(\log x)^5\,dx/x$,

(e) $\dfrac{dx}{x(x^4+1)^{\frac{3}{4}}}$, ($f$) $\dfrac{\sqrt{2\,x-x^2}}{x^2}\operatorname{vers}^{-1}x\,dx$,

(g) $\dfrac{d\theta}{\sqrt{9+5\sec^2\theta}}$, ($h$) $\dfrac{\tan y\,dy}{\sin^2 y}$,

(i) $\dfrac{(3\,x-8)dx}{\sqrt{2\,x^2+5\,x+3}}$, ($j$) $\dfrac{(e^{3x}+5\,e^x)dx}{\sqrt{e^{2x}+e^{-2x}}}$.

23. Simplify this formula for a frictional moment, if $a=1$:

$$M=\frac{4\,\mu ar^2U}{\delta}\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}}\frac{d\phi}{a+\cos\phi}+H\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}}\frac{d\phi}{(a+\cos\phi)^2}.$$

24. Find these integrals, relating to airplane flight:

(a) $\int dy/\sqrt{e^{2ky}-1}$,

(b) $\int_0^T e^{-.065t}(.5\cos .187\,t-.4\sin .187\,t-.5\,e^{-.2t})dt$.

25. How can an economist represent graphically the total utility U of varying quantities, x and y, of two commodities? [If U be plotted vertically, a contour is called an "indifference curve." Why?]

26. Find the area of the paraboloid $z=xy+40$ above a square base, $x=-5$ to $x=5$, $y=-5$ to $y=5$. Use cylindrical coördinates.

27. The same as Ex. 26, using rectangular coördinates.

28. Find the attraction exerted upon a particle of mass M, at $(0, 0, -2)$, by a homogeneous solid bounded by $z=x^2+y^2$ and $z=4$.

29. Simplify this formula for the brightness of the sky:

$$I=4\,al^2\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(\sin^4\phi+\sin^2\phi\cos^2\phi\sin^2\theta)(13-12\cos\phi)\sin\phi\,d\phi\,d\theta.$$

30. In § 84(A), if I is the illumination per sec. per sq. cm., express as a quadruple integral the total illumination from $t=0$ to $t=t_1$ on a portion of the retina, regarded as flat and rectangular. Likewise if circular.

31. Find the total area and volume of a ring generated by revolving a circle of radius 4 in. about an axis 20 in. from the center.

32. A cylinder of height 40 in. and diameter 40 in. is cut by a cone of like height and diameter, whose axis is one element of the cylinder. Find the volume of the cylinder outside the cone.

33. In Ex. 32 find also the moment of inertia, about the element in question, of the outside portion considered, if $D=k$.

34. A thin flat plate is bounded by $x^2=9\,y$ and $y=4$ (feet). Its surface density varies as the distance from the X-axis. Find (a) its mean density, (b) its centroid, (c) the force of water pressure to which it would be subjected (on either face) if submerged vertically with its flat top 2 ft. under water, (d) its perimeter.

35. Find the equation of the cylinder, with axis parallel to OZ, which passes through the intersection of $5\,x+2\,y+10\,z=20$ and $x^2+y^2+z^2=9$. Draw the part of the figure in the first octant.

36. By combining the series for $\log(1+x)$ and $\log(1-x)$, p. 491, show that $\log[(1+x)/(1-x)]=2(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\cdots)$. Take $x=\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$; and calculate the logarithms of 2, 3, 5, 7. Check by tables. How could we calculate also log 9? log 11?

37. Calculate e to 18 decimal places.

38. A formula relating to peptic digestion is $A\log(1-x/A)+x=-Kqt$, where A, K, q are constants. Show that for very small values of x this reduces approximately to $x=\sqrt{2\,Kaqt}$.

39. Integrate $\sinh^5 x\,dx$; also $\sinh x\,dx/\cosh^4 x$.

40. Evaluate: (*a*) $\int_0^\infty x\,e^{-4x}dx$, (*b*) $\int_0^{\frac{\pi}{6}} \frac{\sin x}{x}\,dx$.

41. In throwing 30 dice the probability that the number of "aces" will not differ from 5 by more than 2 is given by the integral

$$\sqrt{\frac{2}{\pi}}\int_0^{.4\sqrt{6}} e^{-\frac{1}{2}t^2}\,dt. \qquad \text{Evaluate this.}$$

42. The mean intensity for certain sound vibrations is

$$\frac{1}{\sqrt{2\,\pi}n}\int_{-\infty}^{\infty} x^2 e^{-x^2/2n^2}\,dx. \qquad \text{Evaluate this.}$$

43. Find the average height of the surface $z=25-x^2-y^2$ above all points of the XY-plane which lie within the surface.

44. Recently in the United States the number of personal incomes (per dollar range in the income I) was approximately: $y=(9\times 10^9)I^{-\frac{9}{4}}$. Find the average income between $I=1600$ and $I=10{,}000$.

45. If all stars were of equal brilliancy at unit distance, and occurred with equal frequency throughout space, would the total illumination at the earth be finite?

46. A certain section of a cylinder is an ellipse, $x=20\sin\phi$, $y=12\cos\phi$. Find its perimeter.

47. The force moving an object varied as in the adjacent table. Find graphically the momentum imparted from $t=0$ to $t=40$. Check by calculation after discovering the most probable formula of the type $F=a+bt+ct^2$.

t	F
0	16
10	8.9
20	3.9
30	1.1
40	0

48. Solve these differential equations [$y' = dy/dx$, etc.]:

(*a*) $(x^3\cos y+e^y)y'=x-3\,x^2\sin y$; (*b*) $(x^2+xy)y'=xy-y^2$;
(*c*) $x^3y'+x^2y=y^2e^x$; (*d*) $y''=60/y^2$.

49. Find the solution of $d^2y/dt^2+4\,dy/dt+13\,y=\sin t$ which, at $t=0$, gives $y=0$ and $dy/dt=10$. Draw a graph of y as a function of t.

50. Find the time required to drain a full hemispherical reservoir, of radius 50 ft., through a hole at the bottom, of radius 1 ft. [The speed of escaping particles (v ft./sec.) varies thus with the depth (x ft.): $v=5\sqrt{x}$.]

51. In using a directive radio, the strength of the wave I varied with the angle θ either side of the axis as in the table. Plot the polar graph; also the "cardioid" $r=a(1+\cos\theta)$. Compare carefully. Also find $\angle\psi$ for each at $\theta=90°$.

θ	I
0	12
30	11
60	9
90	6
120	3
150	1
180	0

52. Find the slope of the epicycloid (28), p. 31, at $\phi=0$.

53. Find the nature of the path of a point P, 2 in. from the center of a circle M, of diameter 20 in., as M rolls internally on a circle of radius 20 in. [Cf. § 22, and Ex. 10 (*b*), p. 373.]

54. Charged particles strike a screen at points $P(x, y)$ given by: $x = eEl^2/2\,mv^2$, $y = eHl^2/2\,mCv$. Find the locus of P for all speeds v.

55. A point moved thus: $r = 5 \sin 10\,t$, $\theta = 5\,t$. Find its path. Also describe its motion fully at the instant $t = \pi/20$.

56. A point moved thus: $x = 8 \cos t$, $y = 5 \sin t$. Find, at $t = \pi/6$: v, τ, a, A, a_t, a_n. Also find the path.

57. Compare the semi-axes of $x^2/16 + y^2/4 = 1$, also the radii of curvature at ends of the axes.

58. Find the area swept over by the radius vector of a comet moving in the curve $r = .8 \sec^2 \theta/2$, from $\theta = 0$ to $\theta = \pi/3$.

59. Find the equations of the line L tangent to the curve $x = 2\,t^2$, $y = t^3$, $z = 12\,t$, at $t = 1$. Where does L meet the plane XOY?

60. Find the angle between the two lines through (3, 2, 1) which lie on the hyperboloid $x^2 + 4\,y^2 - z^2 = 24$.

61. If rays of light emanate from a point on a circular reflector, find their caustic curve after the first reflection.

62. Find the area of the plane $12\,x + 3\,y - 4\,z = 52$ included within the sphere $x^2 + y^2 + z^2 = 25$.

63. Find the evolute of the envelope of the family of lines $y = mx - m^2 - 2$. Draw a figure.

64. By inspection, what sort of curve is represented by $z = 0$ and

(*a*) $2\,x^2 + xy + y^2 = 8$, (*b*) $x^2 + 2\,xy + y^2 + 8\,y = 0$,

(*c*) $5\,x^2 - 2\,xy + 4\,x = 10$, (*d*) $x^2 - 2\,xy + y^2 = 16$?

65. What kind of curve is the locus of $x^2 - 4\,xy - 2\,y^2 - 6\,x + 8 = 0$, $z = 0$? Check by intersections with lines $y = lx$. Also reduce to a standard equation.

66. Plot each following curve, after making any useful tests:

(*a*) $y^3 = 2\,ax^2 - x^3$, (*b*) $x^3 - y^3 - x^2 + 4\,y^2 = 0$.

67. Considering the lowest point of a rolling circle as "instantaneously at rest," show without calculation that

(*a*) The highest point is moving twice as fast as the center;

(*b*) Any point on the circle is instantaneously moving directly toward, or directly away from, the highest point of the circle.

68. By Ex. 67 (*b*) show how to draw a tangent to a cycloid anywhere.

69. The outer corners of two bricks in a wall are (3, 2), (3, 0), (1, −2), (−3, −2), (−3, 0), (−1, 2). Find the equation of the smallest ellipse enclosing these points. What is the inclination of its axis?

70. A sphere of radius r is gently lowered into a glass of water, conical in shape, of radius a and depth h. Express the amount of overflow. (See p. 489.) What r would maximize this?

71. A slip noose is adjusted around a square post, and is tightened by pulling the free end of the rope away along the perpendicular bisector of one side of the post. Ignoring friction find the angle at which the rope leaves the post when in equilibrium.

72. A long sheet of tin 20 in. wide is to be bent into a trough whose cross section is a segment of a circle. What radius will maximize the carrying capacity? Interpret.

73. A uniformly loaded beam of length l in. is supported at its ends. Derive a formula for the bending moment X in. from one end. Show that this is a maximum at the center.

74. A cylindrical tank of diameter 6 ft. has its axis horizontal. When full of water, what is the force of water pressure against one end?

75. In Ex. 74 find also the center of pressure.

76. A wedge is cut from a cylinder of radius 6 in. by a plane through a diameter of the base, inclined 45° to the latter. Determine the centroid of the wedge.

77. Find exactly the following integral, relating to a certain area:

$$\int \frac{(3+5\sin\theta)^4 d\theta}{4\cos\theta+3\sin\theta+5}. \quad \text{[See § 103; or (45)–(46), p. 343.]}$$

78. If $F(y, y') = y\sqrt{1+y'^2}$, simplify the equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$. [Here y' denotes dy/dx.]

79. Solve the differential equation: $1+y'^2-yy''=0$.

80. If the "master reaction" governing bodily growth has the *D.E.*: $dx/dt = kx(a^3-x^3)$, and if $x=\frac{1}{2}a$ at $t=t_1$, find the relation between x and t at any time.

81. The basic differential equation of wave motion is $\partial^2 y/\partial t^2 = a^2\, \partial^2 y/\partial x^2$. Show that this is satisfied by $y=f(x+at)$, where f denotes an arbitrary function. Likewise by $y=f(x-at)$.

82. The cost of a certain cylindrical reservoir, of radius x ft. and height y ft., will be $C=k_1x^2y^2+k_2x^{\frac{11}{6}}+(2.2+\frac{1}{3}\,y)2\,\pi x+\frac{2}{3}\,\pi x^2$, where k_1 and k_2 are constants. Obtain the equation from which we could find the value of x giving the minimum C for a fixed volume. What method would be used to solve that equation?

83. Calculate, if $V=.4$, Fresnel's integrals relating to diffraction:

(*a*) $\displaystyle\int_0^V \sin\left(\tfrac{1}{2}\,\pi v^2\right)dv$, (*b*) $\displaystyle\int_0^V \cos\left(\tfrac{1}{2}\,\pi v^2\right)dv$.

APPENDIX

Symbols and Greek alphabet. Proofs and references. Formulas. Tables of integrals. Numerical tables. Index.

SYMBOLS AND GREEK ALPHABET

$\frac{d}{dx}$, derivative (as to x).

$\frac{\partial}{\partial x}$, partial derivative (as to x).

$\triangle$, triangle.

Δ, increment; finite difference.

$\Delta^{(n)}$, n-th order difference.

$\vec{v}$, directed v (velocity).

$E(k, \phi)$, $F(k, \phi)$, elliptic integrals.

l, m, n, direction cosines.

$\bar{y}$, mean value of y.

gd, Gudermannian.

sin, cos, sine, cosine.

tan, ctn, tangent, cotangent.

sec, csc, secant, cosecant.

cis, cosine $+ i$ sine.

sinh, $\cdots$, hyperbolic sine, etc.

$\sin^{-1}$, $\cdots$, arcsine, etc.

$\int$, integral.

$\rightarrow$, approaches the limit.

$\underset{\Delta x \to 0}{\mathrm{L}}$, limit of, as $\Delta x \to 0$.

∞, infinity.

$\rightarrow \infty$, increases without limit.

$<$, is less than (algebraically).

$>$, is greater than (algebraically).

$\neq$, (is) not equal to.

$1^{(r)}$, radian.

i, imaginary unit $(\sqrt{-1})$.

$n!$, factorial n, $= 1 \cdot 2 \cdot 3 \cdots n$.

e, log base; eccentricity.

θ, polar angle.

ω, angular speed.

α, angular acceleration.

$\log_b$, logarithm of, base b.

D.E., differential equation.

$C_{n,r}(P_{n,r})$, number of combinations (permutations) of n, r at a time.

GREEK ALPHABET

Letter	Name	Letter	Name	Letter	Name	Letter	Name
Α α	Alpha	Η η	Eta	Ν ν	Nu	Τ τ	Tau
Β β	Beta	Θ θ	Theta	Ξ ξ	Xi	Υ υ	Upsilon
Γ γ	Gamma	Ι ι	Iota	Ο ο	Omicron	Φ φ	Phi
Δ δ	Delta	Κ κ	Kappa	Π π	Pi	Χ χ	Chi
Ε ε	Epsilon	Λ λ	Lambda	Ρ ρ	Rho	Ψ ψ	Psi
Ζ ζ	Zeta	Μ μ	Mu	Σ σ	Sigma	Ω ω	Omega

PROOFS AND REFERENCES

INDETERMINATE FORMS

A discussion of all conditions under which the differentiation procedure of §§ 183–84 is applicable lies outside the scope of this course. For rigorous proofs and a discussion of some inadequate common proofs, see J. Pierpont, *The Theory of Functions of Real Variables*, v. 1, pp. 298–312. We merely note conditions sufficient in the basic cases.

(*A*) *For* 0/0 *at* $x=a$. Let $f(x)$, $F(x)$, and their first $(n-1)$ derivatives be zero at $x=a$ and continuous in some interval adjoining (and including) $x=a$. Let $f^{(n)}(x)$ and $F^{(n)}(x)$ be finite in that interval; also continuous and $\neq 0$ at $x=a$. Then

$$\underset{x\to a}{\mathrm{L}}\frac{f(x)}{F(x)}=\frac{f^{(n)}(a)}{F^{(n)}(a)}. \qquad \left[\text{If } n=1, \text{ this is simply } \frac{f'(a)}{F'(a)}.\right]$$

(*B*) *For* 0/0 *at* $x=\infty$. Let $f(x)$ and $F(x)\to 0$ as $x\to\infty$. Let $f'(x)$ and $F'(x)$ be finite and $F'(x)\neq 0$, when x exceeds some value X. Then, if $f'(x)/F'(x)$ approaches a limit L, finite or infinite, $f(x)/F(x)$ approaches L.

(*C*) *For* ∞/∞ *at* $x=a$. Let $f(a)=F(a)=\infty$. Let $f(x)$ and $F(x)$ be continuous, and $f'(x)$ and $F'(x)$ finite, in an interval extending to (not including) $x=a$. Let $f'(x)/F'(x)$ approach some limit L. Then $f(x)/F(x)\to L$.

(*D*) *For* ∞/∞ *at* $x=\infty$. Like (*B*); but $f(\infty)=F(\infty)=\infty$; and $f'(\infty)$, $F'(\infty)$ may $=\infty$.

A DOUBLE INTEGRAL AS A LIMIT OF A SUM

Let $f(x, y)$ be a function continuous within, and on the boundary of, a region R of the XY-plane.* (Fig. 188.)

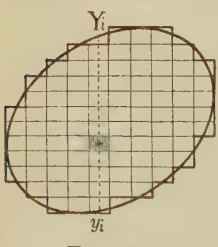


FIG. 188.

Let R be cut into small rectangles $\Delta x\Delta y$, where Δx and Δy are exactly contained in the greatest horizontal and vertical lengths of R. Let n be the number of columns, m_i the number of rectangles in the i-th column which have their centers in R; and $f_j^{(i)}$ the value of $f(x, y)$ at the midpoint of the j-th rectangle in the i-th column. Also let Y_i and y_i be the upper and lower ordinates at the boundary of R, on the vertical line $x=x_i$ bisecting the i-th column.

*That is, $f(x, y)\to f(x_1, y_1)$ if (x, y) approaches (x_1, y_1), both points in R.

Theorem. The limit as $\Delta x \to 0$, *of the limit as* $\Delta y \to 0$, *of the sum of terms* $f_j^{(i)}\,\Delta x\,\Delta y$ for all the rectangles of R, that is,

$$\underset{\Delta x \to 0}{\mathrm{L}}\ \underset{\Delta y \to 0}{\mathrm{L}}\ [(f_1^{(1)}\Delta x\,\Delta y \cdots + f_{m_1}^{(1)}\Delta x\,\Delta y) + \cdots + (f_1^{(n)}\Delta x\,\Delta y \cdots + f_{m_n}^{(n)}\Delta x\,\Delta y)], \qquad (1)$$

is equal to $\int\int f(x, y)\,dy\,dx$, taken over the region R.

Proof. The limit of each parenthesis in (1) as $\Delta y \to 0$, is by § 58, a definite integral; viz.

$$\underset{\Delta y \to 0}{\mathrm{L}}(f_1^{(i)}\Delta y \cdots + f_{mi}^{(i)}\Delta y)\Delta x = \Delta x \int_{y_i}^{Y_i} f(x_i, y)\,dy.$$

As the Y_i and y_i are functions of x_i, we may denote each integral by $F(x_i)$. Then the expression in (1) becomes

$$L = \underset{\Delta x \to 0}{\mathrm{L}}\{F(x_1)\Delta x \cdots + F(x_n)\Delta x\}. \qquad (2)$$

By § 58 again, this limit is the integral $\int F(x)\,dx$, from the smallest to the largest x in R. Thus L is found by integrating $f(x, y)$ twice: (a) with respect to y, while holding x fixed; (b) with respect to x.

$$\therefore \qquad L = \int\int f(x, y)\,dy\,dx, \quad \text{over } R. \qquad (3)$$

Remarks. (I) If the boundary of a region R meets some vertical lines in more than two points, but in a limited number, it can be cut by suitable vertical lines into sub-regions in each of which the theorem holds.

(II) A like theorem can be proved similarly for a continuous function $f(x, y, z)$, and a corresponding three-dimensional region S. Using portions $\Delta x\,\Delta y\,\Delta z$, there would be three limit-steps, giving a triple integral.

ILLUSTRATIONS OF THE FOREGOING THEOREM

(A) *Total Load on a Plane Area R.* If the loading (w lb./sq. ft.) at any point is variable, subdivide R into small rectangles $\Delta x\,\Delta y$ as in Fig. 188. Then the load on the j-th rectangle of the i-th column is approximately $w_j^{(i)}\,\Delta x\,\Delta y$, and the total load is the limit of the sum of all such items. That is, L is the double limit in (1), with w replacing f.

$$\therefore \qquad L = \int\int w\,dy\,dx. \qquad \text{[Cf. §115.]}$$

Remark. We are justified in saying that the load on a rectangle is $w\,\Delta x\,\Delta y$, approx., because w at any point P is *defined* as the limiting value of the average load per sq. ft. on any small area ΔA including P, as $\Delta A \to 0$.

(*B*) *Volume of a Solid Under a Given Surface,* and over a region R of the XY-plane. Let the base R be subdivided into rectangles $\Delta x\, \Delta y$ as formerly. The volume over each is approximately $z_j^{(i)}\, \Delta x\, \Delta y$; and the entire volume is the double limit in (1), with z in place of f. Hence

$$V = \int\int z\, dy\, dx. \qquad \text{[Cf. § 131.]}$$

Remark. The volume of a solid is in general *defined* as the limit of such a sum of rectangular solids. Cf. *Intro.*, § 38.

(*C*) *Area of a Curved Surface S*, over a region R of the XY-plane. This area may be defined as the limit of the sum of plane areas constructed as follows: Let R be subdivided as before, and a tangent plane be drawn at the point of S directly over the midpoint of each rectangle $\Delta x \Delta y$. Let the part of this tangent plane which lies above the rectangle $\Delta x \Delta y$ have an area ΔT, and let γ be its inclination. Then $\Delta T = (\sec \gamma)\, \Delta x \Delta y$; and the total area of S is the double limit in (1), with $\sec \gamma$ in place of f. Hence

$$\text{area } S = \int\int \sec \gamma\, dy\, dx. \qquad \text{[Cf. § 133.]}$$

Remark. The area could be defined otherwise by using portions of the tangent over parts of R, of a different (polar) shape. And the definitions could be proved to be equivalent.

(*D*) *Load on a Plane Area R, in Polar Coördinates.* Subdivide R by concentric circles about some point O, and by radiating lines through O. The quasi-rectangular portion between circles of radii r and $r + \Delta r$ in the i-th sector, of angle $\Delta\theta$, has the area $\frac{1}{2}(r + \Delta r)^2\, \Delta\theta - \frac{1}{2} r^2 \Delta\theta$, or

$$\Delta A = (r\Delta r + \tfrac{1}{2}\,\Delta r^2)\Delta\theta = (r + \tfrac{1}{2}\,\Delta r)\Delta r \Delta\theta.$$

And the load on this portion is approximately $w\Delta A$.

Now $r + \frac{1}{2}\,\Delta r$ is the value of r at the middle of the small portion, say $r_j^{(i)}$. Let m_i be the number of portions whose midpoints fall in R in the i-th sector. Then the total load is the double limit

$$\underset{\Delta\theta \to 0}{\mathrm{L}}\ \underset{\Delta r \to 0}{\mathrm{L}} \Big[\Big(w_1^{(1)} r_1^{(1)}\, \Delta r\Delta\theta \cdots + w_{m_1}^{(1)} r_{m_1}^{(1)}\, \Delta r \Delta\theta\Big) \cdots + \Big(w_1^{(n)} r_1^{(n)} \Delta r \Delta\theta \cdots + w_{m_n}^{(n)} r_{m_n}^{(n)}\, \Delta r \Delta\theta\Big)\Big].$$

This comes under (1), with wr in place of f, and with independent variables r and θ in place of y and x. Hence

$$\text{Load} = \int\int w\, r\, dr\, d\theta. \qquad \text{[Cf. §§116–17.]}$$

Remark. Similar discussions would show that the other integrals set up in Chapter V by using the idea of tiny elements are valid.

SOME STANDARD FORMULAS

(A) ALGEBRA

Roots of Quadratic, $ax^2+bx+c=0$: $x=(-b\pm\sqrt{b^2-4ac})\div 2a$.

(Higher Equations: Synthetic Division, graph, Newton's, or Horner's method.)

Logarithms (Relation of bases e and 10): $\log_e N = 2.30259 \log_{10} N$.

$\log xy = \log x + \log y$, $\log(x/y) = \log x - \log y$,

$\log x^n = n \log x$, $\log \sqrt[n]{x} = \frac{1}{n}\log x$.

Interest (comp. k times per yr.): $A = P(1+r/k)^{kn}$, $P = A \div (1+r/k)^{kn}$.

Definition of e and M: $e = \underset{z\to\infty}{\mathrm{L}}(1+1/z)^z$; $M = \log_{10} e$.

Arithmetical Progression: $l = a+(n-1)d$, $S = \frac{1}{2}n(a+l)$.

Geometrical Progression: $l = ar^{n-1}$, $S = a(r^n-1)\div(r-1)$.

Permutations: $P_{n,r} = n(n-1)(n-2)\cdots(n-r+1)$, $= n!\div(n-r)!$.

Combinations: $C_{n,r} = P_{n,r} \div r! = n! \div [(n-r)!\,r!]$

Probability (of exactly r successes in n independent trials, when the probability of success on 1 trial is p; and $q=1-p$): $p_{n,r} = C_{n,r}p^rq^{n-r}$.

Special values: $C_{n,0}=1$; zero factorial, $0!=1$. [*Intro.*, p. 444]

$e^{-\infty}=0$, $\log 0 = -\infty$; $a^0 = 1\ (a\neq 0)$, $\log 1 = 0\ [a\neq 1, 0]$.

(B) TESTS FOR COMMON LAWS

Power,	$y=kx^n$,	Logarithmic graph straight;
C.I.L.,	$y=Pe^{rx}$,	Semi-logarithmic graph straight;
Linear,	$y=a+bx$,	Ordinary graph straight [cf. next];

Polynomial, $y=a+bx+cx^2\cdots+\gamma x^n$, $\Delta^{(n)}y$ constant if Δx constant.

Simple harmonic, $y=a\cos(kx+\epsilon)=a\sin(kx+\epsilon')$. Graph apparently a sine or cosine curve. Checked by tables.

(C) MENSURATION [See (E) also.]

Circle [For π see p. 501]:	$C=2\pi r$,	$A=\pi r^2$.
Circular sector ($\angle\theta^{(r)}$):	arc $=r\theta$,	$A=\frac{1}{2}r^2\theta$.
Circular segment (''):	chord $=2r\sin(\theta/2)$,	$A=\frac{1}{2}r^2(\theta-\sin\theta)$.
Ellipse: perim. $=4a\,\mathrm{E}(e,\pi/2)$ [p. 499],	$e=\sqrt{a^2-b^2}/a$;	$A=\pi ab$.
Cycloid arch (base $2\pi a$):	length $=8a$,	$A=3\pi a^2$.
Sphere:	$S=4\pi r^2$,	$V=\frac{4}{3}\pi r^3$.
" segment (height h):	$S=2\pi rh$,	$V=\frac{1}{3}\pi h^2(3r-h)$.
Cylinder (right circular):	$S=2\pi rh$,	$V=\pi r^2h$.
Cone (right circular):	$S=\pi rs$,	$V=\frac{1}{3}\pi r^2h$.
Frustum of cone:	$S=\pi(R+r)s$,	$V=\frac{1}{3}\pi h(R^2+Rr+r^2)$.

(D) TRIGONOMETRIC FUNCTIONS

Definition: by coördinates of a point (cf. *Intro.*, § 253–54):

$$\sin\theta=\frac{y}{r},\quad \cos\theta=\frac{x}{r},\quad \tan\theta=\frac{y}{x},\quad \operatorname{ctn}\theta=\frac{x}{y},\quad \sec\theta=\frac{r}{x},\quad \csc\theta=\frac{r}{y}.$$

Also: $\operatorname{vers}\theta=1-\cos\theta$, $\operatorname{covers}\theta=1-\sin\theta$, $\operatorname{exsec}\theta=\sec\theta-1$.

Reciprocals: $\sin\theta\csc\theta=1$, $\cos\theta\sec\theta=1$, $\tan\theta\operatorname{ctn}\theta=1$.

Squares: $\sin^2\theta+\cos^2\theta=1$, $1+\tan^2\theta=\sec^2\theta$, $1+\operatorname{ctn}^2\theta=\csc^2\theta$.

Addition formulas [For $A-B$, change each middle sign in right member].

$$\sin(A+B)=\sin A\cos B+\cos A\sin B,$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B,$$
$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B},\quad \operatorname{ctn}(A+B)=\frac{\operatorname{ctn}A\operatorname{ctn}B-1}{\operatorname{ctn}B+\operatorname{ctn}A}.$$

$\sin(\theta+90°)=\cos\theta$, $\cos(\theta+90°)=-\sin\theta$, $\tan(\theta+90°)=-\operatorname{ctn}\theta$;
$\sin(\theta+180°)=-\sin\theta$, $\cos(\theta+180°)=-\cos\theta$, $\tan(\theta+180°)=\tan\theta$.

Half (or double) $\angle$: $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$, $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$.

Double $\angle$: $\sin 2\theta=2\sin\theta\cos\theta$, $\cos 2\theta=\cos^2\theta-\sin^2\theta$.

Triple $\angle$: $\sin 3\theta=3\sin\theta-4\sin^3\theta$, $\cos 3\theta=4\cos^3\theta-3\cos\theta$.

Sums of functions: For $\sin\theta+\sin\phi$, etc., put $\theta=A+B$, $\phi=A-B$, expand, simplify, and replace A by $\frac{1}{2}(\theta+\phi)$, B by $\frac{1}{2}(\theta-\phi)$.

Products expressed as sums: See (31)–(33), p. 176.

(E) FORMULAS RELATING TO TRIANGLES

Inscribed circle: $r=\sqrt{(h-a)(h-b)(h-c)/h}$, where $h=\frac{1}{2}(a+b+c)$.

Area of triangle: $S=rh$, $=\frac{1}{2}bc\sin A$, $=\frac{1}{2}a^2\sin B\sin C/\sin A$.

Sine law: $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Cosine law: $a^2=b^2+c^2-2bc\cos A$; etc.

Tangent law: $\tan\frac{1}{2}(A-B)=\tan\frac{1}{2}(A+B)[(a-b)/(a+b)]$; etc.

Half-angle law: $\tan\frac{1}{2}A=r/(h-a)$; etc.

(F) HYPERBOLIC FUNCTIONS

Squares: $\cosh^2 x-\sinh^2 x=1$, $1-\tanh^2 x=\operatorname{sech}^2 x$, etc.

Double x: $\sinh 2x=2\sinh x\cosh x$, $\cosh 2x=\cosh^2 x+\sinh^2 x$.

In any trigonometric identity, replace $\sin\theta$ by $i\sinh x$, $\cos\theta$ by $\cosh x$, $\tan\theta$ by $i\tanh x$, etc., where $i=\sqrt{-1}$.

$d\sinh x=\cosh x\,dx$, $d\cosh x=\sinh x\,dx$,
$d\tanh x=\operatorname{sech}^2 x\,dx$, $d\operatorname{ctnh} x=-\operatorname{csch}^2 x\,dx$,
$d\operatorname{sech} x=-\operatorname{sech} x\tanh x\,dx$, $d\operatorname{csch} x=-\operatorname{csch} x\operatorname{ctnh} x\,dx$.

(G) ANALYTIC GEOMETRY

Distance: $D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.
Distance from line (in plane): $d=(aX+bY+c)\div\sqrt{a^2+b^2}$.
Slope of line (in plane): $l=(y_2-y_1)\div(x_2-x_1)$.
Angle between lines: $\tan K_{1,\,2}=(l_2-l_1)\div(1+l_1l_2)$.
Rotation (in plane): $x=x'\cos\phi-y'\sin\phi,\quad y=x'\sin\phi+y'\cos\phi$.
Direction cosines: $l=(x_2-x_1)/D,\quad m=(y_2-y_1)/D,$
$n=(z_2-z_1)/D,\quad l^2+m^2+n^2=1$.
Angle, in space: $\cos A=ll'+mm'+nn'$.
Distance from plane: $d=(aX+bY+cZ+k)\div\sqrt{a^2+b^2+c^2}$.
Division point $(m_1:m_2)$: $x'=(m_1x_2+m_2x_1)\div(m_1+m_2)$; etc.
Coördinate relations: $x=r\cos\theta,\ y=r\sin\theta;\ r=\rho\sin\phi$.
$x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$.
Equations of curves and surfaces: See index references, under name.

(H) TAYLOR AND MACLAURIN SERIES

$$f(x)=f(a)+\frac{f'(a)}{1\,!}(x-a)+\frac{f''(a)}{2\,!}(x-a)^2+\cdots,\qquad E_n<\frac{G_n(x-a)^n}{n\,!},\ \text{num.}$$

$$f(x)=f(0)+\frac{f'(0)}{1\,!}x+\frac{f''(0)}{2\,!}x^2+\cdots,\qquad E_n<\frac{G_nx^n}{n\,!},\quad \text{''}\ .$$

$f(a+x)=f(a)+f'(a)x+f''(a)x^2/2\,!+\cdots$.
$f(x+h)=f(x)+hf'(x)+h^2f''(x)/2\,!+\cdots$.

$e^x=1+x+x^2/2\,!+x^3/3\,!+x^4/4\,!+\cdots,$	any x.
$\sin x=x-x^3/3\,!+x^5/5\,!-x^7/7\,!+\cdots,$	" ".
$\cos x=1-x^2/2\,!+x^4/4\,!-x^6/6\,!+\cdots,$	" ".
$\tan x=x+x^3/3+2\,x^5/15+17\,x^7/315+\cdots,$	$x^2<\pi^2/4$.
$\log(1+x)=x-x^2/2+x^3/3-x^4/4+\cdots,$	$x^2<1$.
$\log(1-x)=-x-x^2/2-x^3/3-x^4/4-\cdots,$	$x^2<1$.
$\sin^{-1}x=x+\frac{x^3}{6}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\cdots,$	$x^2<1$.
$\tan^{-1}x=x-x^3/3+x^5/5-x^7/7+\cdots.$	$x^2<1$.

(I) BINOMIAL THEOREM; FINITE DIFFERENCES

$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1.2}a^{n-2}b^2+\cdots,$ [n, any if $b<a$].

$y_n=f(x+nh)=y_0+n\Delta^{(1)}y+\frac{n(n-1)}{1.2}\Delta^{(2)}y+\cdots$.

(J) DERIVATIVES [See (F), also pp. 38, 46, 55.]

TABLE OF INTEGRALS

In each formula, x may denote any quantity, and dx the differential of that quantity. Cf. also §§ 56, 61–65, 67, 100, 107, 112.

A constant should be added to each integral.

1. $\int x^n\,dx = \frac{1}{n+1}x^{n+1}$ $\qquad (n \neq -1).$

2. $\int \frac{dx}{x} = \log_e x$ $\qquad (= 2.30259 \log_{10} x).$

3. $\int \frac{dx}{(ax+b)(cx+f)} = \frac{1}{af-bc}\log\frac{ax+b}{cx+f}$ $\qquad (af \neq bc).$

4. $\int x\sqrt{ax+b}\,dx = \frac{2}{15\,a^2}(3\,ax-2\,b)(ax+b)^{\frac{3}{2}}.$

5. $\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2}{3\,a^2}(ax-2\,b)\sqrt{ax+b}.$

6. $\int \frac{x\,dx}{(ax+b)^{\frac{3}{2}}} = \frac{2(ax+2\,b)}{a^2\sqrt{ax+b}}.$

7. $\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\log\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}},$ $\qquad$ if $b>0$;

$\qquad = \frac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{ax+b}{-b}},$ $\qquad$ if $b<0$.

8. $\int \frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b}+b\int\frac{dx}{x\sqrt{ax+b}}.$

9. $\int \sqrt{\frac{a+x}{b+x}}\,dx = \sqrt{(a+x)(b+x)}+(a-b)\log(\sqrt{a+x}+\sqrt{b+x}).$

10. $\int \sqrt{\frac{a-x}{b+x}}\,dx = \sqrt{(a-x)(b+x)}+(a+b)\sin^{-1}\sqrt{\frac{b+x}{a+b}}.$

In (9) or (10), every x can be replaced by $-x$, provided the sign of the entire right member be changed.

11. $\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\sin^{-1}\sqrt{\frac{x-a}{b-a}}.$

In the formulas involving $x^2 \pm a^2$, take the upper sign throughout, or else the lower sign throughout, whether all are alike or not.

12. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$

13. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}.$

14. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x+\sqrt{x^2 \pm a^2}) = \begin{cases} \sinh^{-1}\frac{x}{a}, \text{ for } +; \\ \cosh^{-1}\frac{x}{a}, \text{ for } -. \end{cases}$

15. $\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\log(x+\sqrt{x^2 \pm a^2}).$

16. $\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{8}(2x^2 \pm a^2)\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log(x+\sqrt{x^2 \pm a^2}).$

17. $\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\log\frac{a+\sqrt{x^2+a^2}}{x}; \quad \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}.$

18. $\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} - a\log\frac{a+\sqrt{x^2+a^2}}{x}.$

19. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a}.$

20. $\int \frac{dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \pm\frac{x}{a^2\sqrt{x^2 \pm a^2}}; \quad \int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2-x^2}}.$

21. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}.$

22. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

23. $\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}.$

24. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}.$

25. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\log\frac{a+\sqrt{a^2-x^2}}{x}.$

26. $\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\log\frac{a+\sqrt{a^2-x^2}}{x}.$

In (27)–(30), F denotes $\int x^m(ax^n+b)^p\,dx$, and $u=(ax^n+b)$.

27. $$F=\frac{1}{m+np+1}\left[x^{m+1}u^p+npb\int x^m\,u^{p-1}\,dx\right].$$

28. $$F=\frac{1}{bn(p+1)}\left[-x^{m+1}u^{p+1}+(m+n+np+1)\int x^m\,u^{p+1}\,dx\right].$$

29. $$F=\frac{1}{a(m+np+1)}\left[x^{m-n+1}u^{p+1}-b(m-n+1)\int x^{m-n}u^p\,dx\right].$$

30. $$F=\frac{1}{b(m+1)}\left[x^{m+1}u^{p+1}-a(m+n+np+1)\int x^{m+n}u^p\,dx\right].$$

When a denominator is zero, reduce first by another; or rationalize. (§ 97.)

31. $$\int\frac{dx}{(ax^2+b)^p}=\frac{1}{2\,b(p-1)}\,\frac{x}{(ax^2+b)^{p-1}}+\frac{2\,p-3}{2\,b(p-1)}\int\frac{dx}{(ax^2+b)^{p-1}}.$$

32. $$\int\sqrt{2\,ax+x^2}\,dx=\frac{x+a}{2}\sqrt{2\,ax+x^2}-\frac{a^2}{2}\log\left(x+a+\sqrt{2\,ax+x^2}\right).$$

33. $$\int\sqrt{2\,ax-x^2}\,dx=\frac{x-a}{2}\sqrt{2\,ax-x^2}+\frac{a^2}{2}\operatorname{vers}^{-1}\frac{x}{a}.$$

34. $$\int\frac{dx}{\sqrt{2\,ax-x^2}}=\operatorname{vers}^{-1}\frac{x}{a}.$$

35. $$\int\frac{x\,dx}{\sqrt{2\,ax-x^2}}=-\sqrt{2\,ax-x^2}+a\operatorname{vers}^{-1}\frac{x}{a}.$$

36. $$\int\frac{x^m\,dx}{\sqrt{2\,ax-x^2}}=-\frac{x^{m-1}\sqrt{2\,ax-x^2}}{m}+\frac{a(2\,m-1)}{m}\int\frac{x^{m-1}\,dx}{\sqrt{2\,ax-x^2}}.$$

37. $$\int\frac{\sqrt{2\,ax-x^2}\,dx}{x^m}=-\frac{(2\,ax-x^2)^{\frac{3}{2}}}{a(2\,m-3)x^m}+\frac{m-3}{a(2\,m-3)}\int\frac{\sqrt{2\,ax-x^2}}{x^{m-1}}\,dx.$$

38. $$\int x^m\sqrt{2\,ax-x^2}\,dx=-\frac{x^{m-1}(2\,ax-x^2)^{\frac{3}{2}}}{m+2}+\frac{a(2\,m+1)}{m+2}\int x^{m-1}\sqrt{2\,ax-x^2}\,dx.$$

For further forms, see (39)–(49) in the special case, $c=0$.

39. $$\int\frac{dx}{ax^2+bx+c}=\frac{1}{\sqrt{b^2-4\,ac}}\log\frac{2\,ax+b-\sqrt{b^2-4\,ac}}{2\,ax+b+\sqrt{b^2-4\,ac}},\quad\text{if } b^2>4\,ac,$$
$$=\frac{2}{\sqrt{4\,ac-b^2}}\tan^{-1}\frac{2\,ax+b}{\sqrt{4\,ac-b^2}},\quad\text{if } b^2<4\,ac.$$

In (40)–(49), Q denotes (ax^2+bx+c), and $D = b^2-4\,ac$.

40. $$\int\sqrt{Q}\,dx = \frac{(2\,ax+b)\sqrt{Q}}{4\,a} - \frac{D}{8\,a}\int\frac{dx}{\sqrt{Q}}.$$

41. $$\int\frac{dx}{\sqrt{Q}} = \frac{1}{\sqrt{a}}\log\left(\sqrt{Q}+\frac{2\,ax+b}{2\sqrt{a}}\right), \qquad \text{if } a>0,$$
$$= \frac{1}{\sqrt{-a}}\sin^{-1}\frac{-2\,ax-b}{\sqrt{D}}, \qquad \text{if } a<0.$$

42. $$\int\frac{dx}{x\sqrt{Q}} = -\frac{1}{\sqrt{c}}\log\left(\frac{\sqrt{Q}+\sqrt{c}}{x}+\frac{b}{2\sqrt{c}}\right), \qquad \text{if } c>0,$$
$$= \frac{1}{\sqrt{-c}}\sin^{-1}\frac{bx+2\,c}{x\sqrt{D}}, \qquad \text{if } c<0,$$
$$= -\,2\sqrt{Q}/bx, \qquad \text{if } c=0.$$

43. $$\int\frac{dx}{xQ} = \frac{1}{c}\log\frac{x}{\sqrt{Q}} - \frac{b}{2\,c}\int\frac{dx}{Q}.$$ [See (39).]

44. $$\int x^mQ^n\,dx = \frac{1}{a(m+2\,n+1)}\left[x^{m-1}Q^{n+1}-b(m+n)\int x^{m-1}Q^n\,dx - c(m-1)\int x^{m-2}Q^n\,dx\right].$$ (If fails, see § 100.)

45. $$\int x^mQ^n\,dx = \frac{1}{c(m+1)}\left[x^{m+1}Q^{n+1} - b(m+n+2)\int x^{m+1}Q^n\,dx - a(m+2\,n+3)\int x^{m+2}Q^n\,dx\right].$$

46. $$\int x\,Q^n\,dx = \frac{Q^{n+1}}{2\,a(n+1)} - \frac{b}{2\,a}\int Q^n\,dx.$$

47. $$\int\frac{Q^n}{x}\,dx = \frac{Q^n}{2\,n} + c\int\frac{Q^{n-1}}{x}\,dx + \frac{b}{2}\int Q^{n-1}\,dx, \qquad \text{if } n>0,$$
$$= -\frac{Q^{n+1}}{2\,c(n+1)} + \frac{1}{c}\int\frac{Q^{n+1}}{x}\,dx - \frac{b}{2\,c}\int Q^n\,dx, \qquad \text{if } n<0.$$

48. $$\int Q^n\,dx = \frac{(2\,ax+b)Q^n}{2\,a(2\,n+1)} - \frac{nD}{2\,a(2\,n+1)}\int Q^{n-1}\,dx.$$

49. $$\int Q^n\,dx = \frac{(2\,ax+b)Q^{n+1}}{(n+1)D} - \frac{2(2\,n+3)}{(n+1)D}\,a\int Q^{n+1}\,dx.$$

50. $\int \sin x\,dx = -\cos x, \qquad \int \cos x\,dx = \sin x.$

51. $\int \tan x\,dx = -\log \cos x, \qquad \int \operatorname{ctn} x\,dx = \log \sin x.$

52. $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x, \qquad \int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x.$

53. $\int \tan^2 x\,dx = \tan x - x, \qquad \int \operatorname{ctn}^2 x\,dx = -\operatorname{ctn} x - x.$

54. $\int \sec^2 x\,dx = \tan x, \qquad \int \csc^2 x\,dx = -\operatorname{ctn} x.$

55. $\int \sqrt{1-\cos x}\,dx = -2\sqrt{2}\cos\frac{x}{2}, \qquad \int \sqrt{1+\cos x}\,dx = 2\sqrt{2}\sin\frac{x}{2}.$

56. $\int \sec x\,dx = \log(\sec x + \tan x) \qquad [= \operatorname{gd}^{-1} x].$

57. $\int \csc x\,dx = \log(\csc x - \operatorname{ctn} x) \qquad \left[= \log \tan\frac{x}{2}\right].$

58. $\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log(\sec x + \tan x).$

59. $\int \frac{dx}{\sin x \cos x} = \log \tan x.$

Forms like $\sin^n x \cos x\,dx$, $\tan^n x \sec^2 x\,dx$, $\sec^n x\,(\sec x \tan x)\,dx$, are of the type $u^n\,du$.

60. $\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

61. $\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$

62. $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx.$

63. $\int \operatorname{ctn}^n x\,dx = -\frac{\operatorname{ctn}^{n-1} x}{n-1} - \int \operatorname{ctn}^{n-2} x\,dx.$

64. $\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$

65. $\int \csc^n x\,dx = -\frac{\csc^{n-2} x \operatorname{ctn} x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx.$

In (66)–(69), P denotes $\int \sin^m x \cos^n x\,dx$.

66. $P = -\dfrac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \dfrac{m-1}{m+n}\displaystyle\int \sin^{m-2}x\cos^n x\,dx.$

67. $P = \dfrac{\sin^{m+1}x\cos^{n+1}x}{m+1} + \dfrac{m+n+2}{m+1}\displaystyle\int \sin^{m+2}x\cos^n x\,dx.$

68. $P = \dfrac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \dfrac{n-1}{m+n}\displaystyle\int \sin^m x\cos^{n-2}x\,dx.$

69. $P = -\dfrac{\sin^{m+1}x\cos^{n+1}x}{n+1} + \dfrac{m+n+2}{n+1}\displaystyle\int \sin^m x\cos^{n+2}x\,dx.$

70. $\displaystyle\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\tan x\right).$

71. $\displaystyle\int \frac{dx}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2+b^2}}\log\frac{\sqrt{a^2+b^2} - a\cos x + b\sin x}{a\sin x + b\cos x}.$

72. $\displaystyle\int \frac{dx}{a+b\cos x} = -\frac{1}{\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right),$ if $a^2 > b^2$,

$$= \frac{1}{\sqrt{b^2-a^2}}\log\frac{b+a\cos x+\sqrt{b^2-a^2}\sin x}{a+b\cos x}, \text{ if } a^2 < b^2.$$

73. $\displaystyle\int \frac{dx}{(1+l\cos x)^2} = -\frac{l\sin x}{(1-l^2)(1+l\cos x)} + \frac{1}{1-l^2}\int \frac{dx}{(1+l\cos x)}.$

74. $\displaystyle\int \frac{dx}{(1+l\cos x)^3} = \frac{l\sin x(l^2-4-3l\cos x)}{2(1-l^2)^2(1+l\cos x)^2} + \frac{2+l^2}{2(1-l^2)^2}\int \frac{dx}{1+l\cos x}.$

75. $\displaystyle\int \frac{dx}{1+\cos x} = \csc x - \operatorname{ctn} x.$

In (76), (77), S denotes $(a^2 - b^2\sec^2 x)$; $c = \sqrt{a^2-b^2}$.

76. $\displaystyle\int \sqrt{S}\,dx = b\cos^{-1}\left(\frac{b\tan x}{c}\right) - a\cos^{-1}\left(\frac{a\sin x}{c}\right).$

77. $\displaystyle\int S^{\frac{3}{2}}\,dx = \frac{bc^2}{2}\cos^{-1}\left(\frac{b\tan x}{c}\right) - \frac{b^2}{2}\sqrt{S}\tan x + a^2\int \sqrt{S}\,dx.$

78. $\displaystyle\int \sin mx \sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}.$

79. $\displaystyle\int \cos mx \cos nx\,dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}.$ $\quad m \neq n.$

80. $\displaystyle\int \sin mx \cos nx\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}.$

81. $\int \sin^{-1} x\, dx = \left[\frac{\pi}{2}x - \int \cos^{-1} x\, dx\right] = x \sin^{-1} x + \sqrt{1-x^2}.$

82. $\int \tan^{-1} x\, dx = x \tan^{-1} x - \log \sqrt{1+x^2}.$

83. $\int \text{vers}^{-1} x\, dx = (x-1)\,\text{vers}^{-1} x + \sqrt{2x - x^2}.$

84. $\int a^{mx}\, dx = \frac{a^{mx}}{m \log a};$ $\qquad \int e^{mx}\, dx = \frac{e^{mx}}{m}.$

85. $\int x e^{mx}\, dx = e^{mx}\left(\frac{x}{m} - \frac{1}{m^2}\right).$

86. $\int x^n a^{mx}\, dx = \frac{x^n a^{mx}}{m \log a} - \frac{n}{m \log a}\int x^{n-1} a^{mx} dx.$

87. $\int x^n \log x\, dx = \frac{x^{n+1}[(n+1)\log x - 1]}{(n+1)^2}.$

88. $\int e^{kx} \sin nx\, dx = \frac{e^{kx}(k \sin nx - n \cos nx)}{k^2+n^2}.$

89. $\int e^{kx} \cos nx\, dx = \frac{e^{kx}(n \sin nx + k \cos nx)}{k^2+n^2}.$

90. $\int x^m \sin ax\, dx = -\frac{x^m}{a}\cos ax + \frac{m}{a}\int x^{m-1} \cos ax\, dx,$

$$= \frac{x^{m-1}}{a^2}\left(-ax \cos ax + m \sin ax\right) - \frac{m(m-1)}{a^2}\int x^{m-2} \sin ax\, dx.$$

91. $\int x^m \cos ax\, dx = \frac{x^m}{a}\sin ax - \frac{m}{a}\int x^{m-1} \sin ax\, dx,$

$$= \frac{x^{m-1}}{a^2}\left(ax \sin ax + m \cos ax\right) - \frac{m(m-1)}{a^2}\int x^{m-2} \cos ax\, dx.$$

92. $\int \sinh x\, dx = \cosh x;$ $\qquad \int \cosh x\, dx = \sinh x.$

93. $\int \sinh^2 x\, dx = -\frac{1}{2}x + \frac{1}{4}\sinh 2x.$

94. $\int \cosh^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sinh 2x.$

95. $\int \tanh x\, dx = \log \cosh x.$

96. $\int \text{sech}\, x\, dx = 2 \tan^{-1} e^x$ $\qquad [= \text{gd}\, x].$

Other hyperbolic functions: use exponential values or see § 160.

ELLIPTIC INTEGRALS

(I) $$F(k, \phi) = \int_0^\phi \frac{dz}{\sqrt{1-k^2 \sin^2 z}}$$

k	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
	ϕ = radian equivalent of:									
0	.000	.175	.349	.524	.698	.873	1.047	1.222	1.396	1.571
.1	.000	.175	.349	.524	.699	.874	1.049	1.224	1.400	1.575
.2	.000	.175	.349	.525	.700	.877	1.054	1.232	1.410	1.588
.3	.000	.175	.350	.526	.703	.882	1.062	1.244	1.427	1.610
.4	.000	.175	.350	.527	.707	.889	1.074	1.262	1.452	1.643
.5	.000	.175	.351	.529	.712	.898	1.090	1.285	1.485	1.686
.6	.000	.175	.352	.532	.718	.911	1.112	1.320	1.534	1.752
.7	.000	.175	.353	.536	.727	.928	1.142	1.370	1.608	1.854
.8	.000	.175	.354	.539	.736	.947	1.178	1.431	1.705	1.993
.9	.000	.175	.355	.544	.748	.974	1.233	1.534	1.885	2.275
.95	.000	.175	.356	.546	.755	.991	1.270	1.602	2.057	2.601
.99	.000	.175	.356	.549	.761	1.006	1.306	1.705	2.313	3.425
1.00	.000	.175	.356	.549	.763	1.011	1.317	1.735	2.436	∞

(II) $$E(k, \phi) = \int_0^\phi \sqrt{1-k^2 \sin^2 z}\, dz$$

k	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
	ϕ = radian equivalent of:									
0	.000	.175	.349	.524	.698	.873	1.047	1.222	1.396	1.571
.1	.000	.175	.349	.523	.698	.872	1.046	1.219	1.393	1.566
.2	.000	.174	.349	.523	.696	.869	1.041	1.212	1.383	1.554
.3	.000	.174	.348	.521	.693	.864	1.032	1.200	1.367	1.533
.4	.000	.174	.348	.520	.690	.857	1.021	1.184	1.344	1.504
.5	.000	.174	.347	.518	.685	.848	1.008	1.163	1.316	1.467
.6	.000	.174	.347	.515	.679	.837	.989	1.135	1.277	1.417
.7	.000	.174	.346	.512	.672	.823	.965	1.099	1.227	1.351
.8	.000	.174	.345	.509	.664	.808	.940	1.060	1.172	1.278
.9	.000	.174	.343	.505	.654	.789	.907	1.008	1.095	1.173
.95	.000	.174	.342	.502	.649	.778	.888	.976	1.046	1.103
.99	.000	.174	.342	.501	.644	.769	.871	.948	.999	1.021
1.00	.000	.174	.342	.500	.643	.766	.866	.940	.985	1.000

THE INTEGRAL $I_n = \int_0^{\frac{\pi}{2}} \sin^n z\, dz = \int_0^{\frac{\pi}{2}} \cos^n z\, dz$

(a) $I_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots \quad n} \dfrac{\pi}{2}$, if n is a positive even integer;

(b) $I_n = \dfrac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots \quad n}$, if n is a positive odd integer.

HYPERBOLIC FUNCTIONS

x	e^x	e^{-x}	sinh x	cosh x	tanh x	gd x	Equiv.	
0	1.0000	1.00000	.0000	1.0000	.0000	.0000(r)	0°	0′
.1	1.1052	.90484	.1002	1.0050	.0997	.0998	5	43
.2	1.2214	.81873	.2013	1.0201	.1974	.1987	11	23
.3	1.3499	.74082	.3045	1.0453	.2913	.2956	16	56
.4	1.4918	.67032	.4108	1.0811	.3799	.3897	22	20
.5	1.6487	.60653	.5211	1.1276	.4621	.4804	27	31
.6	1.8221	.54881	.6367	1.1855	.5370	.5669	32	29
.7	2.0138	.49659	.7586	1.2552	.6044	.6490	37	11
.8	2.2255	.44933	.8881	1.3374	.6640	.7262	41	37
.9	2.4596	.40657	1.0265	1.4331	.7163	.7985	45	45
1.0	2.7183	.36788	1.1752	1.5431	.7616	.8658	49	36
1.1	3.0042	.33287	1.3356	1.6685	.8005	.9281	53	11
1.2	3.3201	.30119	1.5095	1.8107	.8337	.9857	56	29
1.3	3.6693	.27253	1.6984	1.9709	.8617	1.0387	59	31
1.4	4.0552	.24660	1.9043	2.1509	.8854	1.0872	62	18
1.5	4.4817	.22313	2.1293	2.3524	.9051	1.1317	64	51
1.6	4.9530	.20190	2.3756	2.5775	.9217	1.1724	67	10
1.7	5.4739	.18268	2.6456	2.8283	.9354	1.2094	69	18
1.8	6.0496	.16530	2.9422	3.1075	.9468	1.2432	71	14
1.9	6.6859	.14957	3.2682	3.4177	.9562	1.2739	72	59
2.0	7.3891	.13534	3.6269	3.7622	.9640	1.3018	74	35
2.5	12.1825	.08208	6.0502	6.1323	.9866	1.4070	80	37
3.0	20.0855	.04979	10.0179	10.0677	.9951	1.4713	84	18
3.5	33.1155	.03020	16.5426	16.5728	.9982	1.5104	86	32
4.0	54.5982	.01832	27.2899	27.3082	.9993	1.5342	87	54

Some Further Powers of *e*

(See also pages 502–503)

x	e^x	e^{-x}
2.1	8.1662	.12246
2.2	9.0250	.11080
2.3	9.9742	.10026
2.4	11.0232	.09072
2.5	12.1825	.08208
2.6	13.4637	.07427
2.7	14.8797	.06721
2.8	16.4446	.06081
2.9	18.1741	.05502
3.0	20.0855	.04979

x	e^x	e^{-x}
3.1	22.198	.045049
3.2	24.533	.040762
3.3	27.113	.036883
3.4	29.964	.033373
3.5	33.115	.030197
3.6	36.598	.027324
3.7	40.447	.024724
3.8	44.701	.022371
3.9	49.402	.020242
4.0	54.598	.018316

x	e^x	e^{-x}
4.5	90.017	.011109
5.0	148.41	.006738
5.5	244.69	.004087
6.0	403.43	.002479
6.5	665.14	.001503
7.0	1096.6	.000912
7.5	1808.0	.000553
8.0	2981.0	.000335
8.5	4914.8	.000203
9.0	8103.1	.000123

PROBABILITY INTEGRAL: $P(X) = \frac{1}{\sqrt{2\pi}} \int_0^X e^{-\frac{1}{2}x^2}\,dx$

WITH ORDINATES OF THE CURVE $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

X	P	y
0	.0000	.3989
.1	.0398	.3969
.2	.0793	.3910
.3	.1179	.3814
.4	.1554	.3683
.5	.1915	.3521
.6	.2258	.3332
.7	.2580	.3123
.8	.2881	.2897
.9	.3159	.2661
1.0	.3413	.2420

X	P	y
1.0	.3413	.2420
1.1	.3643	.2178
1.2	.3849	.1942
1.3	.4032	.1714
1.4	.4192	.1497
1.5	.4332	.1295
1.6	.4452	.1109
1.7	.4554	.0940
1.8	.4641	.0789
1.9	.4713	.0656
2.0	.4773	.0540

X	P	y
2.0	.4773	.0540
2.2	.4861	.0355
2.4	.4918	.0224
2.6	.4953	.0136
2.8	.4974	.0079
3.0	.4986	.0044
3.2	.4993	.0024
3.4	.4997	.0012
3.6	.4998	.0006
3.8	.4999	.0003
4.0	.5000	.0001

GAMMA FUNCTION: $\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx$

p	.00	.05	.10	.15	.20	.25	.30	.35	.40	.45
$\Gamma(p+1)$	1.0000	.9735	.9513	.9330	.9181	.9064	.8960	.8913	.8874	.8857
log	.0000	9.9883	9.9783	9.9699	9.9629	9.9573	9.9530	9.9500	9.9481	9.9473

p	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95
$\Gamma(p+1)$	.8861	.8888	.8935	.9001	.9087	.9190	.9313	.9456	.9618	.9795
log	9.9475	9.9488	9.9511	9.9543	9.9584	9.9633	9.9691	9.9757	9.9831	9.9912

When p is a positive integer, $\Gamma(p+1) = p!$ In other cases where p exceeds 1, use the reduction formula, $\Gamma(p+1) = p\Gamma(p)$, until a value of p between 0 and 1 is reached.

IMPORTANT CONSTANTS

Symbol	Value	Log_{10}
π	3.141 5927	.497 1499
$1/\pi$	.318 3099	9.502 8501
$1/\sqrt{2\pi}$	.398 9423	9.600 9100
$\pi/180$	.017 4533	8.241 8774
$180/\pi$	57.295 780	1.758 1226

Symbol	Value	Log_{10}
e	2.718 2818	.434 2945
$\log_{10} e$	.434 2945	9.637 7843
$\log_e 10$	2.302 5851	.362 2157
g	980.599*	2.991 492
g	32.1722†	1.507 478

* cm./sec². † ft./sec². These are for sea level, latitude 45°. For any other latitude $L°$, multiply by $(1 - .0026 \cos 2L)$.

NATURAL LOGARITHMS (Base e)

N	0		1	2	3	4	5	6	7	8	9
1.0	0.0	000	100	198	296	392	488	583	677	770	862
1.1		953	*044	*133	*222	*310	*398	*484	*570	*655	*740
1.2	0.1	823	906	989	*070	*151	*231	*311	*390	*469	*546
1.3	0.2	624	700	776	852	927	*001	*075	*148	*221	*293
1.4	0.3	365	436	507	577	646	716	784	853	920	988
1.5	0.4	055	121	187	253	318	383	447	511	574	637
1.6		700	762	824	886	947	*008	*068	*128	*188	*247
1.7	0.5	306	365	423	481	539	596	653	710	766	822
1.8		878	933	988	*043	*098	*152	*206	*259	*313	*366
1.9	0.6	419	471	523	575	627	678	729	780	831	881
2.0		931	981	*031	*080	*129	*178	*227	*275	*324	*372
2.1	0.7	419	467	514	561	608	655	701	747	793	839
2.2		885	930	975	*020	*065	*109	*154	*198	*242	*286
2.3	0.8	329	372	416	459	502	544	587	629	671	713
2.4		755	796	838	879	920	961	*002	*042	*083	*123
2.5	0.9	163	203	243	282	322	361	400	439	478	517
2.6		555	594	632	670	708	746	783	821	858	895
2.7		933	969	*006	*043	*080	*116	*152	*188	*225	*260
2.8	1.0	296	332	367	403	438	473	508	543	578	613
2.9		647	682	716	750	784	818	852	886	919	953
3.0		986	*019	*053	*086	*119	*151	*184	*217	*249	*282
3.1	1.1	314	346	378	410	442	474	506	537	569	600
3.2		632	663	694	725	756	787	817	848	878	909
3.3		939	969	*000	*030	*060	*090	*119	*149	*179	*208
3.4	1.2	238	267	296	326	355	384	413	442	470	499
3.5		528	556	585	613	641	669	698	726	754	782
3.6		809	837	865	892	920	947	975	*002	*029	*056
3.7	1.3	083	110	137	164	191	218	244	271	297	324
3.8		350	376	402	429	455	481	507	533	558	584
3.9		610	635	661	686	712	737	762	788	813	838
4.0		863	888	913	938	962	987	*012	*036	*061	*085
4.1	1.4	110	134	159	183	207	231	255	279	303	327
4.2		351	375	398	422	446	469	493	516	540	563
4.3		586	609	633	656	679	702	725	748	770	793
4.4		816	839	861	884	907	929	951	974	996	*019
4.5	1.5	041	063	085	107	129	151	173	195	217	239
4.6		261	282	304	326	347	369	390	412	433	454
4.7		476	497	518	539	560	581	602	623	644	655
4.8		686	707	728	748	769	790	810	831	851	872
4.9		892	913	933	953	974	994	*014	*034	*054	*074
5.0	1.6	094	114	134	154	174	194	214	233	253	273

e^x

x	e^x
.00	1.0000
.02	1.0202
.04	1.0408
.06	1.0618
.08	1.0833
.10	1.1052
.12	1.1275
.14	1.1503
.16	1.1735
.18	1.1972
.20	1.2214
.22	1.2461
.24	1.2712
.26	1.2969
.28	1.3231
.30	1.3499
.32	1.3771
.34	1.4049
.36	1.4333
.38	1.4623
.40	1.4918
.42	1.5220
.44	1.5527
.46	1.5841
.48	1.6161
.50	1.6487
.52	1.6820
.54	1.7160
.56	1.7507
.58	1.7860
.60	1.8221
.62	1.8589
.64	1.8965
.66	1.9348
.68	1.9739
.70	2.0138
.72	2.0544
.74	2.0959
.76	2.1383
.78	2.1815
.80	2.2255

Larger or smaller numbers: add (+ or −) multiple of log 10

Ex. I. $1720 = 1.72 \times 10^3$. $\therefore$ log 1720 = log 1.72 + 3 log 10.

Ex. II. $.0172 = 1.72 \times 10^{-2}$. $\therefore$ log .0172 = log 1.72 − 2 log 10.

Further Values: See p. 500, or locate values of x among *logs* in the main table, and read e^x from **N**-column.

MULTIPLES OF $Log_e 10$

log 10 = 2.3026	4 log 10 = 9.2103	− log 10 = .6974 − 3
2 log 10 = 4.6052	5 log 10 = 11.5129	−2 log 10 = .3948 − 5
3 log 10 = 6.9078	6 log 10 = 13.8155	−3 log 10 = .0922 − 7

NATURAL LOGARITHMS (Base *e*)

N	0	1	2	3	4	5	6	7	8	9
5.0	1.6 094	114	134	154	174	194	214	233	253	273
5.1	292	312	332	351	371	390	409	429	448	467
5.2	487	506	525	544	563	582	601	620	639	658
5.3	677	696	715	734	752	771	790	808	827	845
5.4	864	882	901	919	938	956	974	993	*011	*029
5.5	1.7 047	066	084	102	120	138	156	174	192	210
5.6	228	246	263	281	299	317	334	352	370	387
5.7	405	422	440	457	475	492	509	527	544	561
5.8	579	596	613	630	647	664	681	699	716	733
5.9	750	766	783	800	817	834	851	867	884	901
6.0	918	934	951	967	984	*001	*017	*034	*050	*066
6.1	1.8 083	099	116	132	148	165	181	197	213	229
6.2	245	262	278	294	310	326	342	358	374	390
6.3	405	421	437	453	469	485	500	516	532	547
6.4	563	579	594	610	625	641	656	672	687	703
6.5	718	733	749	764	779	795	810	825	840	856
6.6	871	886	901	916	931	946	961	976	991	*006
6.7	1.9 021	036	051	066	081	095	110	125	140	155
6.8	169	184	199	213	228	242	257	272	286	300
6.9	315	330	344	359	373	387	402	416	430	444
7.0	459	473	488	502	516	530	544	559	573	587
7.1	601	615	629	643	657	671	685	699	713	727
7.2	741	755	769	782	796	810	824	838	851	865
7.3	879	892	906	920	933	947	961	974	988	*001
7.4	2.0 015	028	042	055	069	082	096	109	122	136
7.5	149	162	176	189	202	215	229	242	255	268
7.6	281	295	308	321	334	347	360	373	386	399
7.7	412	425	438	451	464	477	490	503	516	528
7.8	541	554	567	580	592	605	618	631	643	656
7.9	669	681	694	707	719	732	744	757	769	782
8.0	794	807	819	832	844	857	869	882	894	906
8.1	919	931	943	956	968	980	992	*005	*017	*029
8.2	2.1 041	054	066	080	090	102	114	126	138	150
8.3	163	175	187	199	211	223	235	247	258	270
8.4	282	294	306	318	330	342	353	365	377	389
8.5	401	412	424	436	448	460	471	483	494	506
8.6	518	529	541	552	564	576	587	599	610	622
8.7	633	645	656	668	679	691	702	713	725	736
8.8	748	759	770	782	793	804	815	827	838	849
8.9	861	872	883	894	905	917	928	939	950	961
9.0	972	983	994	*006	*017	*028	*039	*050	*061	*072
9.1	2.2 083	094	105	116	127	137	148	159	170	181
9.2	192	203	214	225	235	246	257	268	279	289
9.3	300	311	322	332	343	354	364	375	386	396
9.4	407	418	428	439	450	460	471	481	492	502
9.5	513	523	534	544	555	565	576	586	597	607
9.6	618	628	638	649	659	670	680	690	701	711
9.7	721	732	742	752	762	773	783	793	803	814
9.8	824	834	844	854	865	875	885	895	905	915
9.9	925	935	946	956	966	976	986	996	*006	*016
10	2.3 026	036	046	056	066	076	086	096	106	115

e^{-x}

x	e^{-x}
.00	1.0000
.02	.9802
.04	.9608
.06	.9418
.08	.9231
.10	.9048
.12	.8869
.14	.8694
.16	.8521
.18	.8353
.20	.8187
.22	.8025
.24	.7866
.26	.7711
.28	.7558
.30	.7408
.32	.7261
.34	.7118
.36	.6977
.38	.6839
.40	.6703
.42	.6570
.44	.6440
.46	.6313
.48	.6188
.50	.6065
.52	.5945
.54	.5827
.56	.5712
.58	.5599
.60	.5488
.62	.5379
.64	.5273
.66	.5169
.68	.5066
.70	.4966
.72	.4868
.74	.4771
.76	.4677
.78	.4584
.80	.4493

Further values: See p. 500, or use

$$e^{-x} = \frac{1}{e^x}.$$

See **N**-column for e^x values, x being in body of Table.

Trigonometric Functions (Radian Measure)

$\theta^{(r)}$	$\sin\theta$	$\cos\theta$	$\tan\theta$
.00	.000	1.000	.000
.05	.050	.999	.050
.10	.100	.995	.100
.15	.149	.989	.151
.20	.199	.980	.203
.25	.247	.969	.255
.30	.296	.955	.309
.35	.343	.939	.365
.40	.389	.921	.423
.45	.435	.900	.483
.50	.479	.878	.546
.60	.565	.825	.684
.70	.644	.765	.842
.80	.717	.697	1.030
.90	.783	.622	1.260

$\theta^{(r)}$	$\sin\theta$	$\cos\theta$	$\tan\theta$
1.0	.841	.540	1.557
1.5	.997	.071	14.101
2.0	.909	− .416	− 2.185
2.5	.598	− .801	− .747
3.0	.141	− .990	− .143
3.5	− .351	− .936	.375
4.0	− .757	− .654	1.158
4.5	− .978	− .211	4.637
5.0	− .959	.284	− 3.379
5.5	− .706	.709	− .996
6.0	− .279	.960	− .291
7.0	.657	.754	.871
8.0	.989	− .146	− 6.800
9.0	.412	− .911	− .452
10.	− .544	− .839	.648

Radians to Degrees ($1^{(r)} = 57° 17' 44''.806$)

	Radians			Tenths			Hundredths			Thousandths			Ten-Thousandths		Hundred-Thousandths
	°	′	″	°	′	″	°	′	″	°	′	″	′	″	″
1	57	17	45	5	43	46	0	34	23	0	3	26	0	21	2
2	114	35	30	11	27	33	1	8	45	0	6	53	0	41	4
3	171	53	14	17	11	19	1	43	08	0	10	19	1	02	6
4	229	10	59	22	55	06	2	17	31	0	13	45	1	22	8
5	286	28	44	28	38	52	2	51	53	0	17	11	1	43	10
6	343	46	29	34	22	39	3	26	16	0	20	38	2	04	12
7	401	4	14	40	6	25	4	0	38	0	24	04	2	24	14
8	458	21	58	45	50	12	4	35	01	0	27	30	2	45	16
9	515	39	43	51	33	58	5	9	24	0	30	56	3	06	19

Degrees to Radians ($1° = .017453293^{(r)}$)

1°	.01745	**10°**	.17453	**1′**	.00029	**10′**	.00291	**1″**	.000005
2°	.03491	**20°**	.34907	**2′**	.00058	**15′**	.00436	**2″**	.000010
3°	.05236	**30°**	.52360	**3′**	.00087	**20′**	.00582	**3″**	.000015
4°	.06981	**40°**	.69813	**4′**	.00116	**25′**	.00727	**4″**	.000019
5°	.08727	**50°**	.87266	**5′**	.00145	**30′**	.00873	**5″**	.000024
6°	.10472	**60°**	1 04720	**6′**	.00175	**35′**	.01018	**6″**	.000029
7°	.12217	**70°**	1.22173	**7′**	.00204	**40′**	.01164	**7″**	.000034
8°	.13963	**80°**	1.39626	**8′**	.00233	**45′**	.01309	**8″**	.000039
9°	.15708	**90°**	1.57080	**9′**	.00262	**50′**	.01454	**9″**	.000044

TRIGONOMETRIC FUNCTIONS
and their common logarithms

Angle	Sine Value	Sine log	Tangent Value	Tangent log	Cotangent Value	Cotangent log	Cosine Value	Cosine log	
0°	.0000	.....	.0000	.....	.....	.....	1.0000	0.0000	90°
1°	.0175	8.2419	.0175	8.2419	57.290	1.7581	.9998	9.9999	89°
2°	.0349	8.5428	.0349	8.5431	28.636	1.4569	.9994	9.9997	88°
3°	.0523	8.7188	.0524	8.7194	19.081	1.2806	.9986	9.9994	87°
4°	.0698	8.8436	.0699	8.8446	14.301	1.1554	.9976	9.9989	86°
5°	.0872	8.9403	.0875	8.9420	11.430	1.0580	.9962	9.9983	85°
6°	.1045	9.0192	.1051	9.0216	9.5144	0.9784	.9945	9.9976	84°
7°	.1219	9.0859	.1228	9.0891	8.1443	0.9109	.9925	9.9968	83°
8°	.1392	9.1436	.1405	9.1478	7.1154	0.8522	.9903	9.9958	82°
9°	.1564	9.1943	.1584	9.1997	6.3138	0.8003	.9877	9.9946	81°
10°	.1736	9.2397	.1763	9.2463	5.6713	0.7537	.9848	9.9934	80°
11°	.1908	9.2806	.1944	9.2887	5.1446	0.7113	.9816	9.9919	79°
12°	.2079	9.3179	.2126	9.3275	4.7046	0.6725	.9781	9.9904	78°
13°	.2250	9.3521	.2309	9.3634	4.3315	0.6366	.9744	9.9887	77°
14°	.2419	9.3837	.2493	9.3968	4.0108	0.6032	.9703	9.9869	76°
15°	.2588	9.4130	.2679	9.4281	3.7321	0.5719	.9659	9.9849	75°
16°	.2756	9.4403	.2867	9.4575	3.4874	0.5425	.9613	9.9828	74°
17°	.2924	9.4659	.3057	9.4853	3.2709	0.5147	.9563	9.9806	73°
18°	.3090	9.4900	.3249	9.5118	3.0777	0.4882	.9511	9.9782	72°
19°	.3256	9.5126	.3443	9.5370	2.9042	0.4630	.9455	9.9757	71°
20°	.3420	9.5341	.3640	9.5611	2.7475	0.4389	.9397	9.9730	70°
21°	.3584	9.5543	.3839	9.5842	2.6051	0.4158	.9336	9.9702	69°
22°	.3746	9.5736	.4040	9.6064	2.4751	0.3936	.9272	9.9672	68°
23°	.3907	9.5919	.4245	9.6279	2.3559	0.3721	.9205	9.9640	67°
24°	.4067	9.6093	.4452	9.6486	2.2460	0.3514	.9135	9.9607	66°
25°	.4226	9.6259	.4663	9.6687	2.1445	0.3313	.9063	9.9573	65°
26°	.4384	9.6418	.4877	9.6882	2.0503	0.3118	.8988	9.9537	64°
27°	.4540	9.6570	.5095	9.7072	1.9626	0.2928	.8910	9.9499	63°
28°	.4695	9.6716	.5317	9.7257	1.8807	0.2743	.8829	9.9459	62°
29°	.4848	9.6856	.5543	9.7438	1.8040	0.2562	.8746	9.9418	61°
30°	.5000	9.6990	.5774	9.7614	1.7321	0.2386	.8660	9.9375	60°
31°	.5150	9.7118	.6009	9.7788	1.6643	0.2212	.8572	9.9331	59°
32°	.5299	9.7242	.6249	9.7958	1.6003	0.2042	.8480	9.9284	58°
33°	.5446	9.7361	.6494	9.8125	1.5399	0.1875	.8387	9.9236	57°
34°	.5592	9.7476	.6745	9.8290	1.4826	0.1710	.8290	9.9186	56°
35°	.5736	9.7586	.7002	9.8452	1.4281	0.1548	.8192	9.9134	55°
36°	.5878	9.7692	.7265	9.8613	1.3764	0.1387	.8090	9.9080	54°
37°	.6018	9.7795	.7536	9.8771	1.3270	0.1229	.7986	9.9023	53°
38°	.6157	9.7893	.7813	9.8928	1.2799	0.1072	.7880	9.8965	52°
39°	.6293	9.7989	.8098	9.9084	1.2349	0.0916	.7771	9.8905	51°
40°	.6428	9.8081	.8391	9.9238	1.1918	0.0762	.7660	9.8843	50°
41°	.6561	9.8169	.8693	9.9392	1.1504	0.0608	.7547	9.8778	49°
42°	.6691	9.8255	.9004	9.9544	1.1106	0.0456	.7431	9.8711	48°
43°	.6820	9.8338	.9325	9.9697	1.0724	0.0303	.7314	9.8641	47°
44°	.6947	9.8418	.9657	9.9848	1.0355	0.0152	.7193	9.8569	46°
45°	.7071	9.8495	1.0000	0.0000	1.0000	0.0000	.7071	9.8495	45°
	Cosine Value	Cosine log	Cotangent Value	Cotangent log	Tangent Value	Tangent log	Sine Value	Sine log	Angle

Note: log sec $x = -$ log cos x, log csc $x = -$ log sin x.

COMMON LOGARITHMS (Base 10)

N	0	1	2	3	4	5	6	7	8	9	u. d.
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4.2
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	3.8
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3.5
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3.2
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3.0
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	2.8
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	2.6
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2.5
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2.4
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2.2
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2.1
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2.0
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	1.9
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	1.8
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	1.8
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	1.7
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	1.6
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	1.6
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	1.5
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1.5
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1.4
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1.4
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1.3
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1.3
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1.3
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1.2
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1.2
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1.2
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1.1
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1.1
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1.1
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1.0
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1.0
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1.0
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1.0
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1.0
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	.9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	.9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	.9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	.9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	.9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	.8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	.8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	.8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	.8

Note: The column u. d. (= unit difference) may be used in interpolating. Multiply the *u. d.* value by figure in 4th place of given number and add to logarithm read from table for first 3 figures of number.

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9	*u. d.*
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	.8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	.8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	.8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	.7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	.7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	.7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	.7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	.7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	.7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	.7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	.7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	.7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	.6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	.6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	.6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	.6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	.6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	.6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	.6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	.6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	.6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	.6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	.6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	.6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	.5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	.5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	.5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	.5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	.5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	.5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	.5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	.5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	.5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	.5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	.5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	.5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	.5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	.5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	.5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	.5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	.5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	.5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	.4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	.4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	.4

Characteristics

Ex. I. $N = 7.65$ (between 1 and 10): $\log 7.65 = 0 +$ decimal.
Ex. II. $N = 76500$ $(= 7.65 \times 10^4)$: $\log 76500 = 4 +$ decimal.
Ex. III. $N = .0765$ $(= 7.65 \times 10^{-2})$: $\log .0765 =$ decimal $- 2$.

SQUARE ROOTS

N	$\sqrt{N}$	$\sqrt{10N}$	N	$\sqrt{N}$	$\sqrt{10N}$	N	$\sqrt{N}$	$\sqrt{10N}$
1.0	1.0000	3.1623	**4.0**	2.0000	6.3246	**7.0**	2.6457	8.3666
1.1	1.0488	3.3166	**4.1**	2.0248	6.4031	**7.1**	2.6646	8.4261
1.2	1.0954	3.4641	**4.2**	2.0494	6.4807	**7.2**	2.6833	8.4853
1.3	1.1402	3.6055	**4.3**	2.0736	6.5574	**7.3**	2.7019	8.5440
1.4	1.1832	3.7417	**4.4**	2.0976	6.6332	**7.4**	2.7203	8.6023
1.5	1.2247	3.8730	**4.5**	2.1213	6.7082	**7.5**	2.7386	8.6603
1.6	1.2649	4.0000	**4.6**	2.1448	6.7823	**7.6**	2.7568	8.7178
1.7	1.3038	4.1231	**4.7**	2.1679	6.8557	**7.7**	2.7749	8.7750
1.8	1.3416	4.2426	**4.8**	2.1909	6.9282	**7.8**	2.7928	8.8318
1.9	1.3784	4.3589	**4.9**	2.2136	7.0000	**7.9**	2.8107	8.8882
2.0	1.4142	4.4721	**5.0**	2.2361	7.0711	**8.0**	2.8284	8.9443
2.1	1.4491	4.5826	**5.1**	2.2583	7.1414	**8.1**	2.8460	9.0000
2.2	1.4832	4.6904	**5.2**	2.2804	7.2111	**8.2**	2.8636	9.0554
2.3	1.5166	4.7958	**5.3**	2.3022	7.2801	**8.3**	2.8810	9.1104
2.4	1.5492	4.8990	**5.4**	2.3238	7.3485	**8.4**	2.8983	9.1652
2.5	1.5811	5.0000	**5.5**	2.3452	7.4162	**8.5**	2.9155	9.2195
2.6	1.6125	5.0990	**5.6**	2.3664	7.4833	**8.6**	2.9326	9.2736
2.7	1.6432	5.1962	**5.7**	2.3875	7.5498	**8.7**	2.9496	9.3274
2.8	1.6733	5.2915	**5.8**	2.4083	7.6158	**8.8**	2.9665	9.3808
2.9	1.7029	5.3852	**5.9**	2.4290	7.6811	**8.9**	2.9833	9.4340
3.0	1.7321	5.4772	**6.0**	2.4495	7.7460	**9.0**	3.0000	9.4868
3.1	1.7607	5.5678	**6.1**	2.4698	7.8102	**9.1**	3.0166	9.5394
3.2	1.7889	5.6569	**6.2**	2.4900	7.8740	**9.2**	3.0332	9.5917
3.3	1.8166	5.7446	**6.3**	2.5100	7.9373	**9.3**	3.0496	9.6437
3.4	1.8439	5.8310	**6.4**	2.5298	8.0000	**9.4**	3.0659	9.6954
3.5	1.8708	5.9161	**6.5**	2.5495	8.0623	**9.5**	3.0822	9.7468
3.6	1.8974	6.0000	**6.6**	2.5690	8.1240	**9.6**	3.0984	9.7980
3.7	1.9235	6.0828	**6.7**	2.5884	8.1854	**9.7**	3.1145	9.8489
3.8	1.9494	6.1644	**6.8**	2.6077	8.2462	**9.8**	3.1305	9.8995
3.9	1.9748	6.2450	**6.9**	2.6268	8.3066	**9.9**	3.1464	9.9499
4.0	2.0000	6.3246	**7.0**	2.6458	8.3666	**10.0**	3.1623	10.0000

RECIPROCALS
of numbers between 1.0 and 9.9

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	1.0000	.9091	.8333	.7692	.7143	.6667	.6250	.5882	.5556	.5263
2	5000	4762	4545	4348	4167	4000	3846	3704	3571	3448
3	3333	3226	3125	3030	2941	2857	2778	2703	2632	2564
4	2500	2439	2381	2326	2273	2222	2174	2128	2083	2041
5	2000	1996	1923	1887	1852	1818	1786	1754	1724	1695
6	1667	1639	1613	1587	1563	1538	1515	1493	1471	1449
7	1429	1408	1389	1370	1351	1333	1316	1299	1282	1266
8	1250	1235	1220	1205	1190	1176	1163	1149	1136	1124
9	1111	1099	1087	1075	1064	1053	1042	1031	1020	1010

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Further Applications to Various Fields, Chiefly in the Exercises

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Advanced mathematics for engineers. (2nd ed.) H. W. Reddick and F. H. Miller. New York: John Wiley; London: Chapman & Hall, 194. Pp. xii + 5087. (Illustrated.) $5.00.

This book, written by two professors at Cooper Union Institute of Technology in New York, is the result of lectures for juniors and seniors in engineering courses at this Institute. The first edition was published in 1938. This second edition shows that the efforts of the authors have met with the appreciation of their colleagues.

The 11 chapters deal with ordinary differential equations, hyperbolic functions, elliptic integrals, infinite series, Fourier series, Gamma and Bessel functions, partial derivatives and partial differential equations, vector analysis, probability, functions of complex variables, and the operational calculus. This constitutes a considerable amount of material, and it is treated very well. A large number of problems, the answers to which are given in the last pages of the book, enrich the text.

Some interesting applications enliven the theory. Examples of the use of hyperbolic functions are Schiele's pivot and the buckling of a rotating shaft; elliptic functions are illustrated by the skipping rope and the field intensity due to a circular current. The problem of a wire clamped at its lower end at a small angle with the vertical serves as a case which leads to Bessel functions. As applications of partial differential equations of the second order we meet the determination of the resultant magnetic field strength arising from a current excited in a solenoid and the induced eddy currents set up in the copper core, as well as a problem cf the drying of a porous slab by evaporation. The chapter on the operational calculus includes not only the Heaviside operators, but also Bromwich' line integrals.

The second edition differs from the first mainly by an appendix of 5 pages on "Units and Dimensional Analysis," a considerable extension of the number of problems, and the addition of a few new topics. The authors mention specifically among these new problems a mechanical brake problem involving elliptic integrals (based on an article by I. Opatowksi. *Amer. math. Monthly*, 1941, p. 443), a more extended treatment of Fourier series, and an article on the vibrating membrane in a rigid rectangular frame.

This reviewer is very well pleased with the authors' clear way of exposition; it is a pleasure to read this book. It can be strongly recommended to students of engineering and physics who are in need of advanced calculus either as their main text or as supplementary reading. It ranks with the best books on this subject in any language.

D. J. Struik

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